Towards Capturing Order-Independent P

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Descriptive Complexity

Query				**Answer**
$q_1\ q_2\ \cdots\ q_n$	$\mapsto$	**Computation**	$\mapsto$	$a_1\ a_2\ \cdots\ a_i\ \cdots\ a_m$

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Restrict attention to the complexity of computing individual bits of the output, i.e., **decision problems.**

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How rich a language do we need to **express** property S?

There is a **computable isomorphism** between these two approaches.

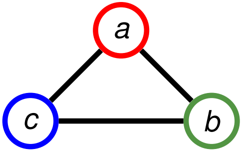
Think of the Input as a Finite Logical Structure

$H \quad = \quad (\{a, b, c\}, \leq, E^H, R^H, G^H, B^H)$

Colored

Graph

H



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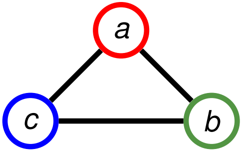

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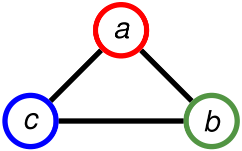

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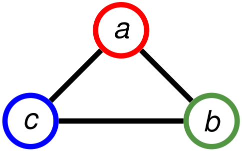

First-Order Logic

input symbols: $E, R, Y, B, \ldots$
variables: $x, y, z, \ldots$
boolean connectives: $\wedge, \vee, \neg$
quantifiers: $\forall, \exists$
numeric symbols: $=, \leq, +, \times, min, max$

$$\alpha \equiv \forall x \exists y\ E(x, y)$$

$$\beta \equiv \forall xy\ (\neg E(x, x)\ \wedge\ (E(x, y) \rightarrow E(y, x)))$$

$$\gamma \equiv \forall x\, ((\forall y\ x \leq y)\ \rightarrow\ R(x))$$

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It is **easy** to test if input, H, satisfies α $(H \models \alpha)$.

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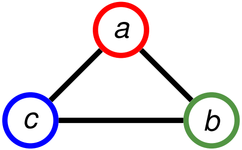
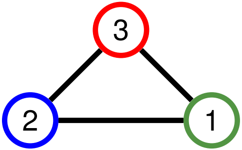


$G \quad 1 \leq 2 \leq 3$



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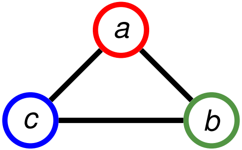

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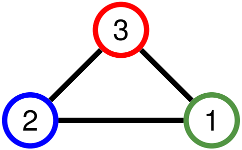

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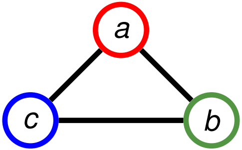

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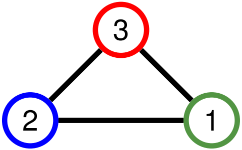

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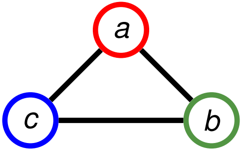

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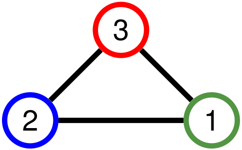

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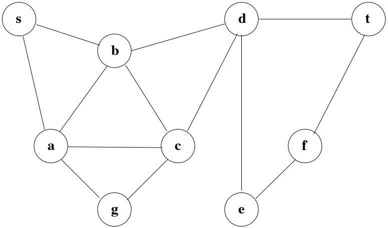
α and β are **order independent**; γ is **order dependent**

Second-Order Logic: FO plus Relation Variables

$$\Phi_{\text{3color}} \equiv \exists R^1 G^1 B^1 \forall x\, y\, ((R(x) \vee G(x) \vee B(x)) \wedge (E(x,y) \rightarrow (\neg(R(x) \wedge R(y)) \wedge \neg(G(x) \wedge G(y)) \wedge \neg(B(x) \wedge B(y)))))$$



Second-Order Logic: FO plus Relation Variables

Fagin's Theorem: NP $=$ SO$\exists$

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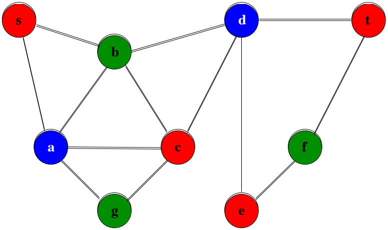

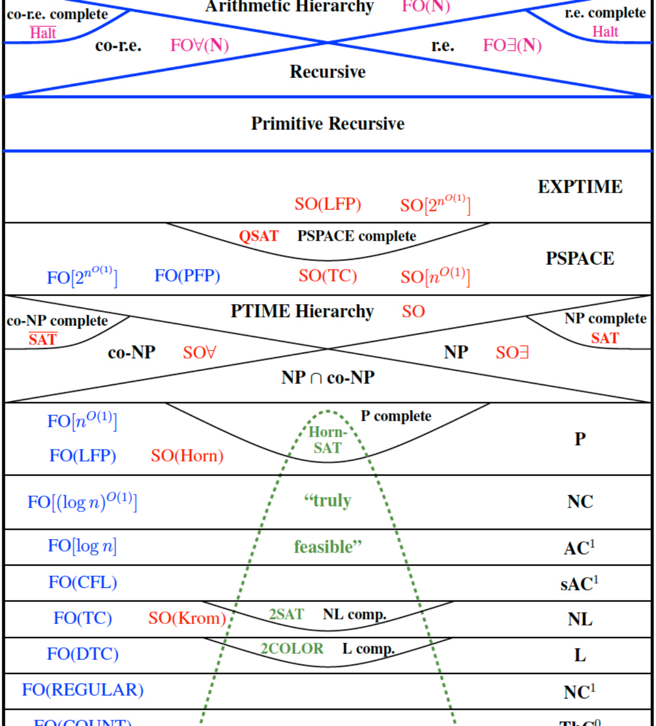

Arithmetic Hierarchy
FO(N)
co-r.e. complete
Halt
r.e. complete
Halt
co-r.e.
FO∀(N)
r.e.
FO∃(N)
Recursive
Primitive Recursive
EXPTIME
SO(LFP)
SO[$2^{n^{O(1)}}$]
QSAT
PSPACE complete
PSPACE
FO[$2^{n^{O(1)}}$]
FO(PFP)
SO(TC)
SO[$n^{O(1)}$]
PTIME Hierarchy
SO
co-NP complete
SAT
NP complete
SAT
co-NP
SO∀
NP
SO∃
NP ∩ co-NP
FO[$n^{O(1)}$]
P complete
Horn-SAT
FO(LFP)
SO(Horn)
P
FO[$(\log n)^{O(1)}$]
"truly
NC
FO[$\log n$]
feasible"
AC^1
FO(CFL)
sAC^1
FO(TC)
SO(Krom)
2SAT
NL comp.
NL
FO(DTC)
2COLOR
L comp.
L
FO(REGULAR)
NC^1
FO(COUNT)
ThC^0

Inductive Definitions and Least Fixed Point

$$\text{REACH} \;=\; \{G, s, t \;|\; s \xrightarrow{\star} t\}$$

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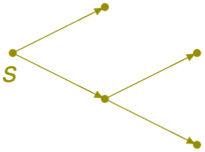

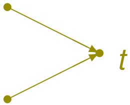

Inductive Definitions and Least Fixed Point

$$E^\star(x,y) \stackrel{\text{def}}{=} x = y \vee E(x,y) \vee \exists z(E^\star(x,z) \wedge E^\star(z,y))$$

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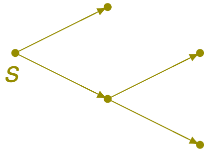

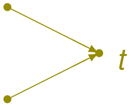

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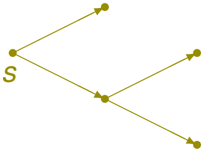

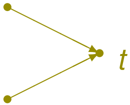

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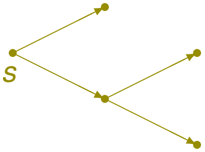

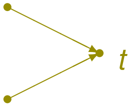

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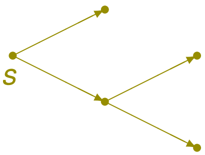

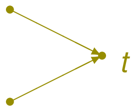

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LFP is a Polynomial Iteration Operator

Thm. $\mathrm{P} = \mathrm{FO(LFP)} = \mathrm{FO}[n^{O(1)}]$

$\mathrm{FO}[n^{O(1)}]$ means for graphs with n vertices, the formula φ_n expressing the property has $n^{O(1)}$ quantifiers, but only a **fixed number** of requantified **variables**, $x_1, \ldots, x_k$, i.e, $\varphi_n \in \mathcal{L}^k$.

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Wanted: a language capturing Order-Independent P (**OIP**).

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$\mathrm{FO}_{(\mathrm{wo}\leq)}\mathrm{(LFP)} \; \subseteq$ **OIP**

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Want to Capture Order-Independent P (OIP)

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How do we prove EVEN $\notin \mathrm{FO(wo{\leq})(LFP)}$?

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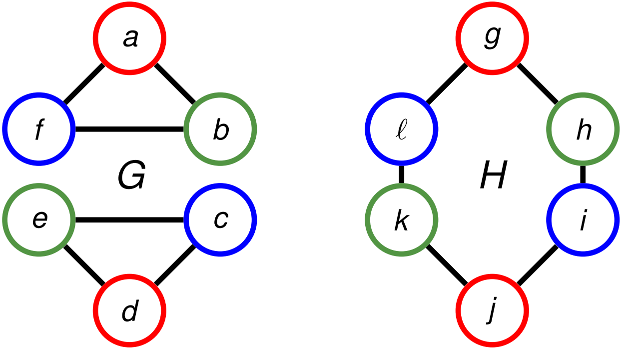
$\mathcal{G}_m^k(G, H)$ m moves, k pebbles, 2 players



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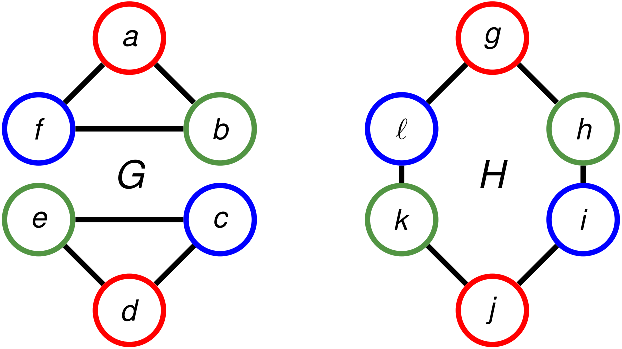

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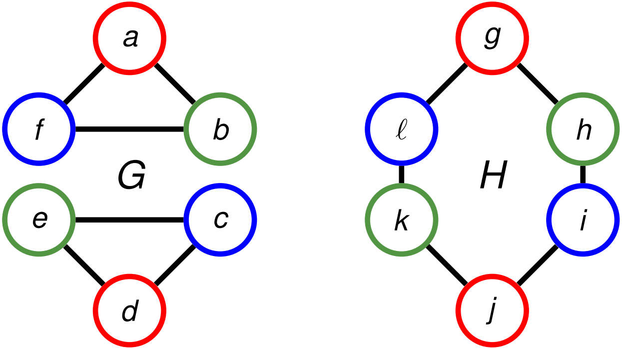

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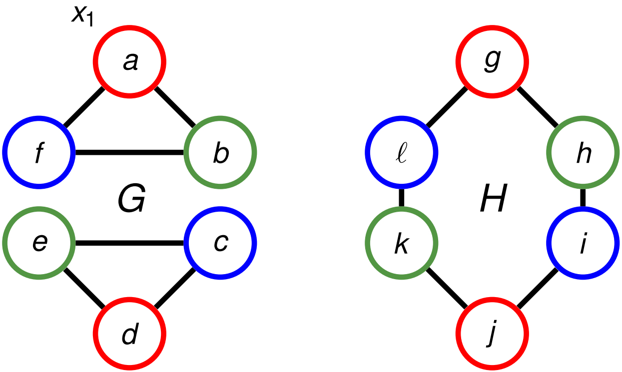

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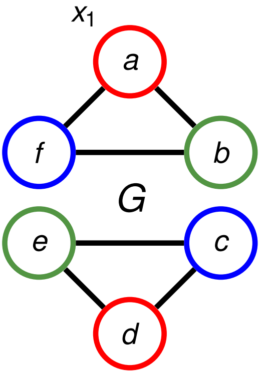

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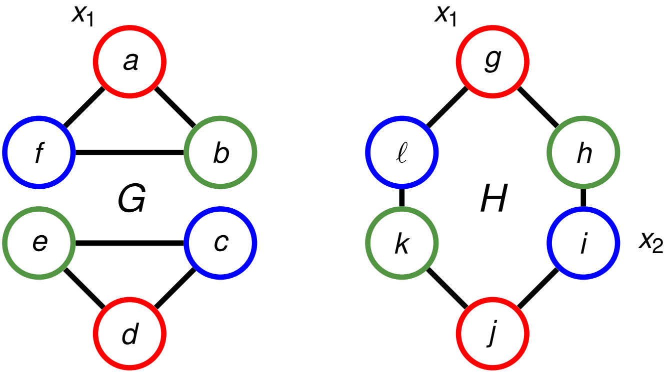

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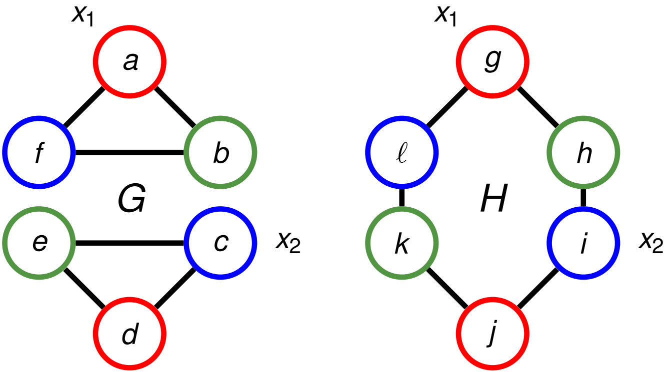

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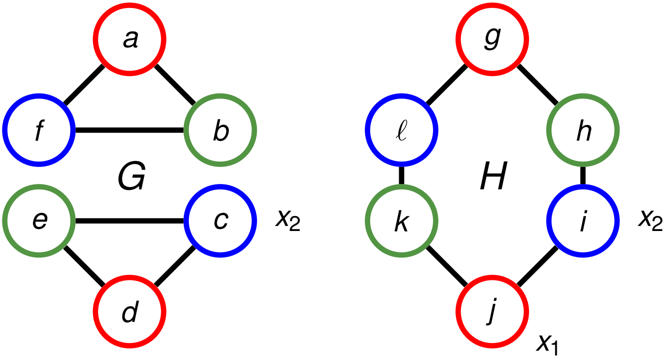

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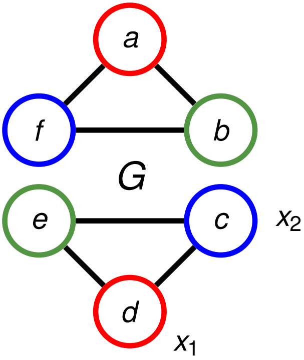

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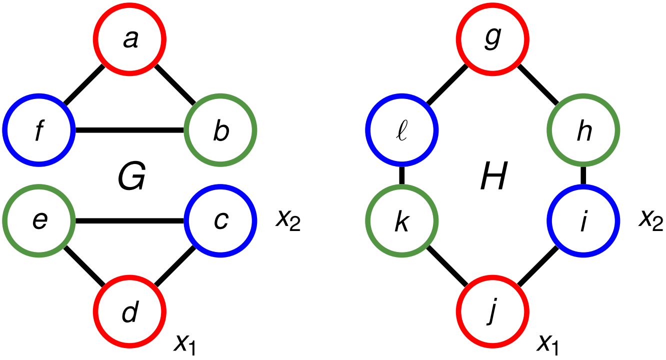

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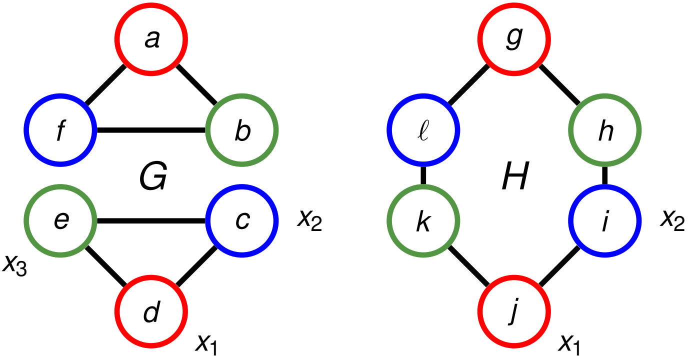

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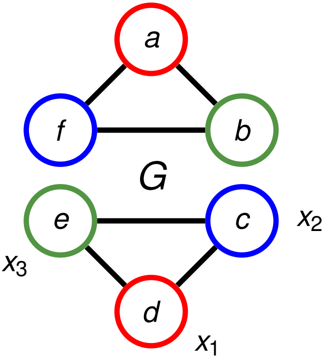

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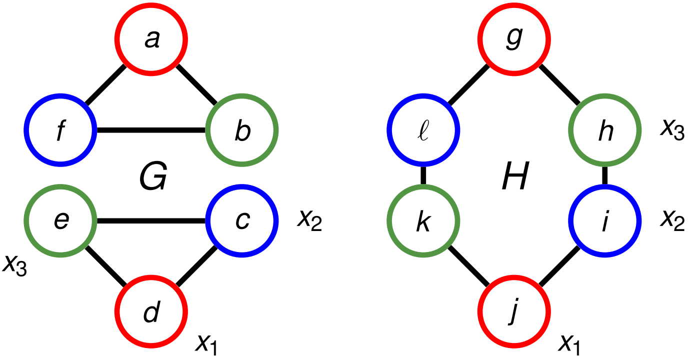

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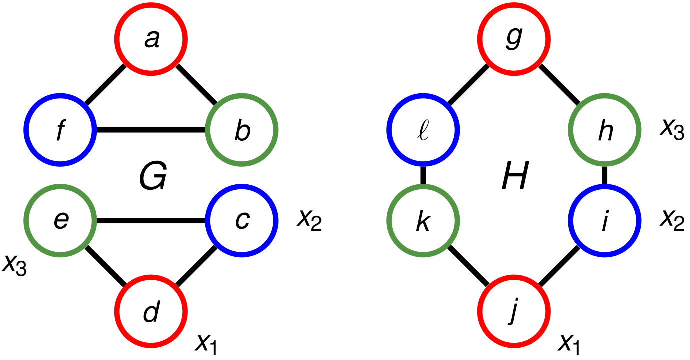
For all m, **D** wins $\mathcal{G}_m^2(G, H)$; but **S** wins $\mathcal{G}_3^3(G, H)$.



Fundamental Thm of Ehrenfeucht-Fraïssé Games

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Thm. **D** has a winning strategy on the m-move, k-pebble game on G, H iff G and H agree on all formulas using k variables and quantifier depth m.

$$G \sim^k_m H \quad \Leftrightarrow \quad G \equiv^k_m H$$

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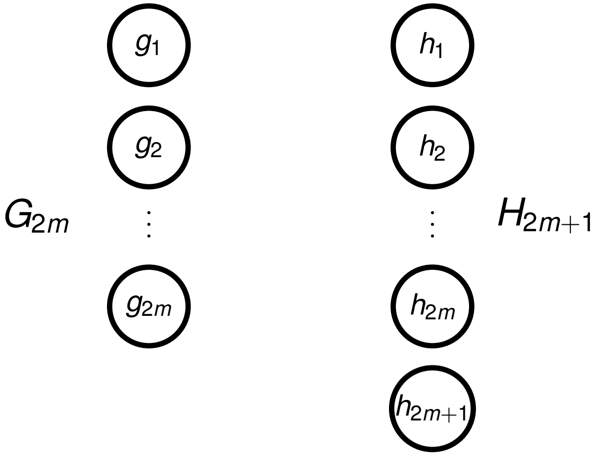

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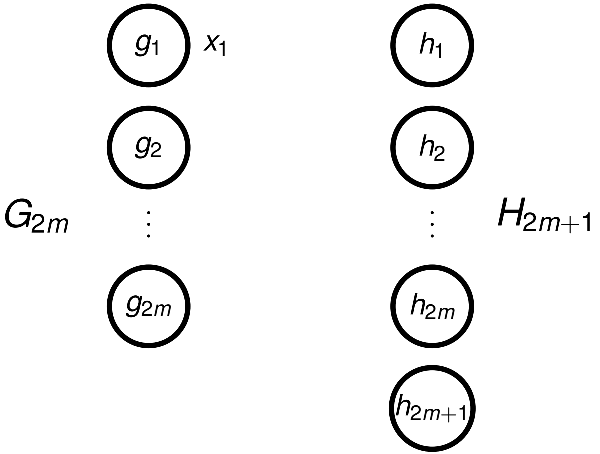

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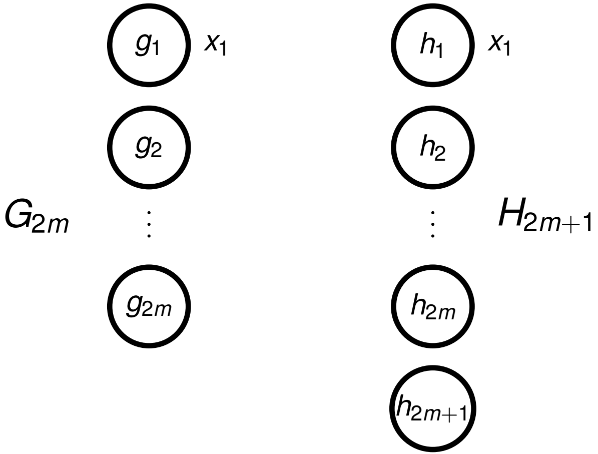

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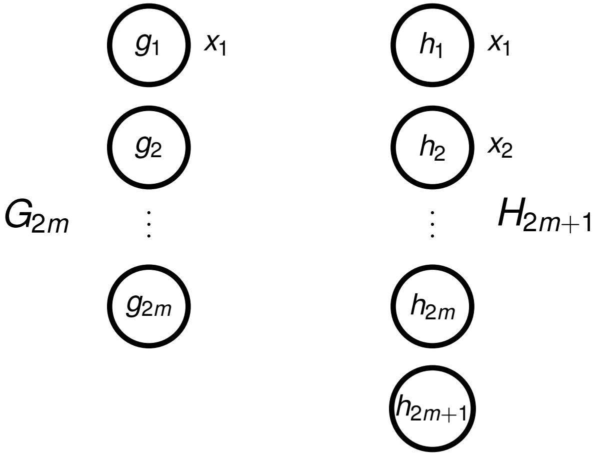

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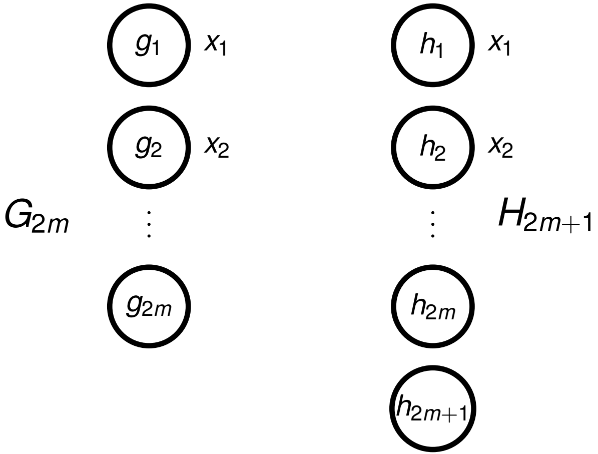

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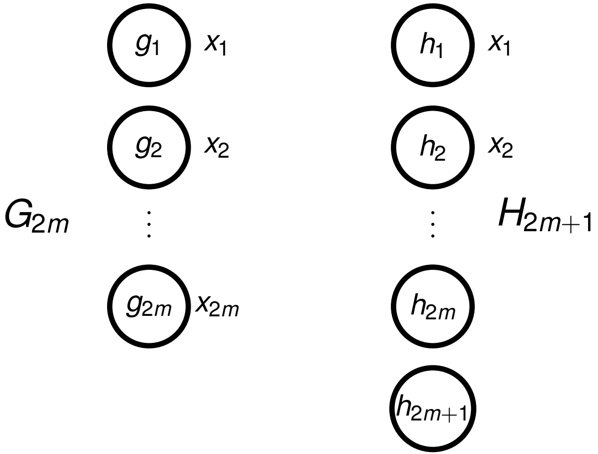

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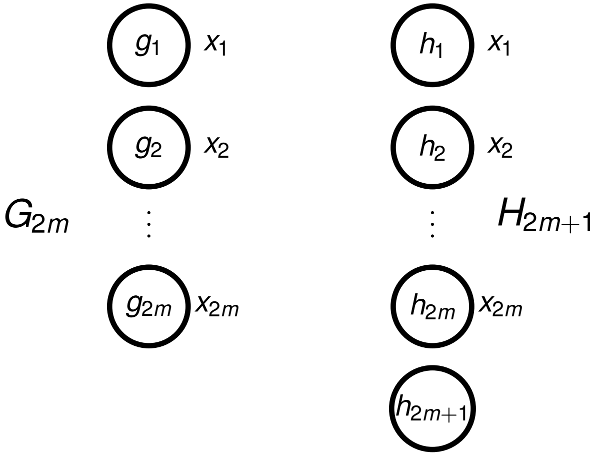

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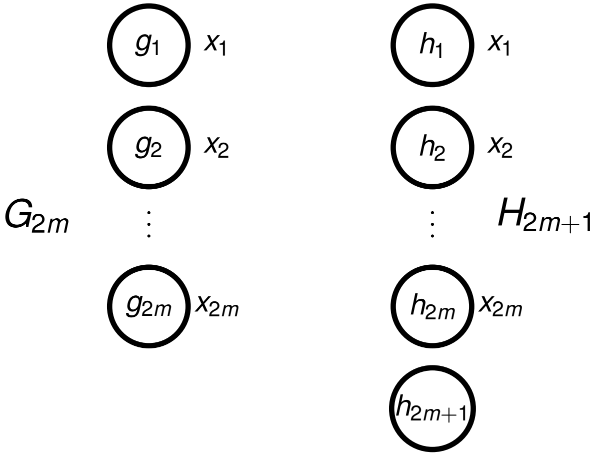

$$G_{2m} \sim^{2m} H_{2m+1}$$

□

Add Counting to FO Logic

Two sorts: **Numbers**: $\{0, 1, \ldots, n\}$, $\leq$, Plus, Times and
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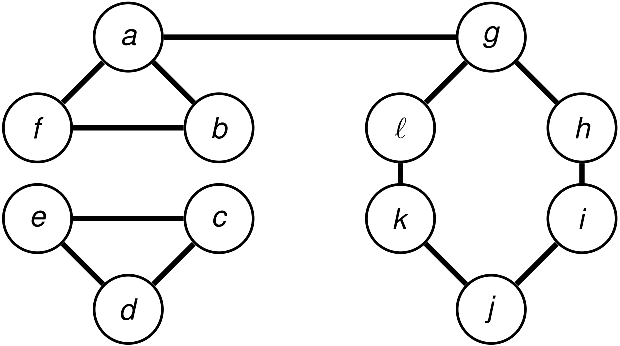
Stable Coloring of Vertices

Start with a colored graph, and repeatedly color each vertex by how many neighbors it has of each color.



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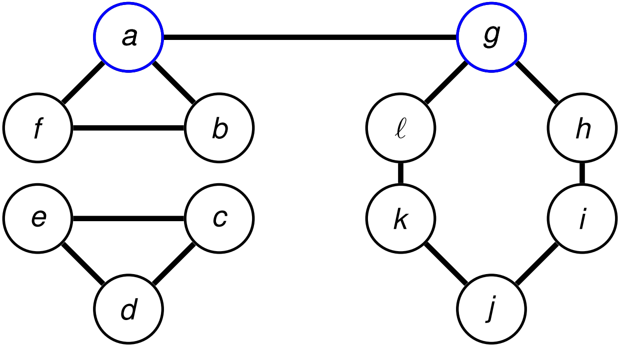

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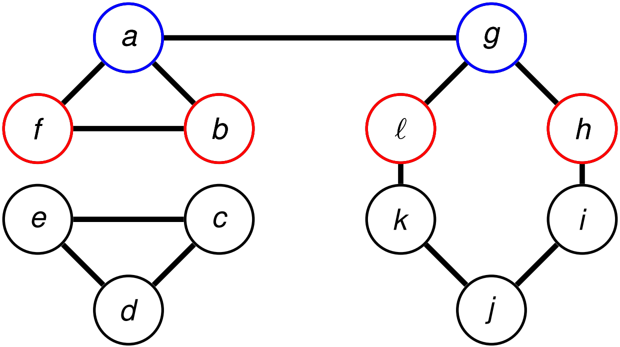

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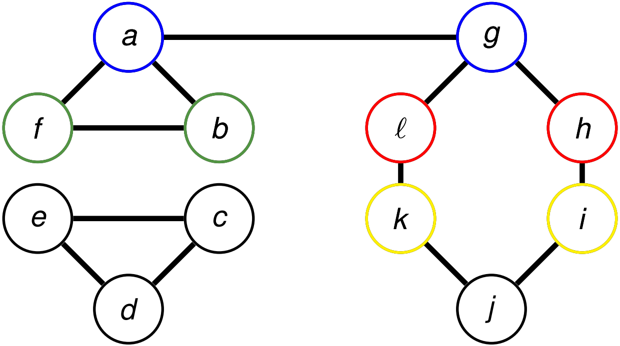

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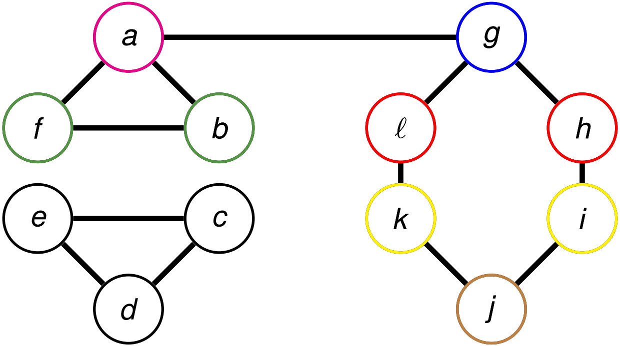

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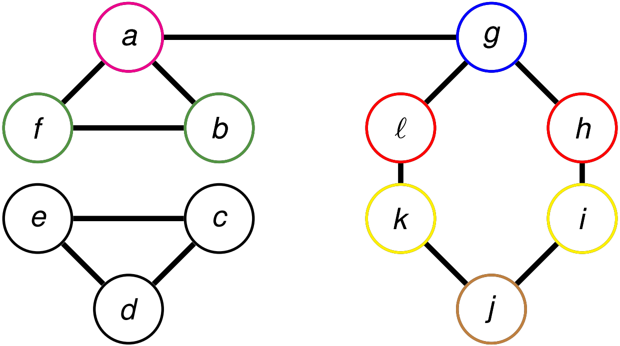

Thm. Stable Coloring of Vertices $=$ C^2 type.

Round m of stable coloring is quantifier depth of C^2 formula.

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Thm. [Babai, Erdos, Selkow] With high probability, after four iterations of stable coloring, each vertex of a random graph has a unique color, i.e., the C_4^2-type of each vertex is unique.

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Thm. [Babai, 2015] $\quad \mathrm{GI} \in \mathrm{DTIME}[n^{\log^7 n}]$. (Before this it was only known that $\mathrm{GI} \in \mathrm{DTIME}[n^{\sqrt{n}}]$.)

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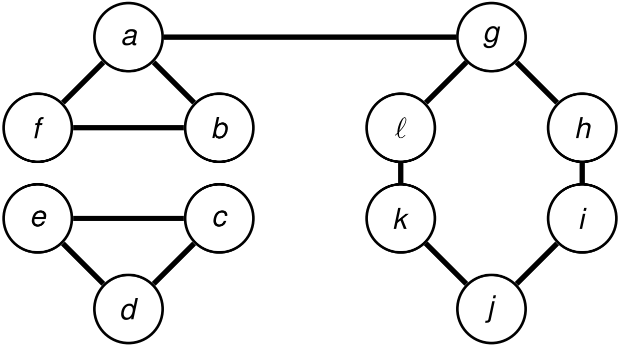
proof: Apply arbitrary $\mathrm{FO(LFP)}$ formula to the canonical form of the input graph. □

Particularizing Means Uniquely Coloring Some Vertex



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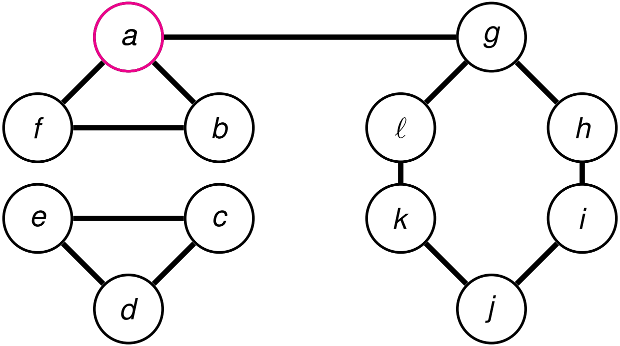

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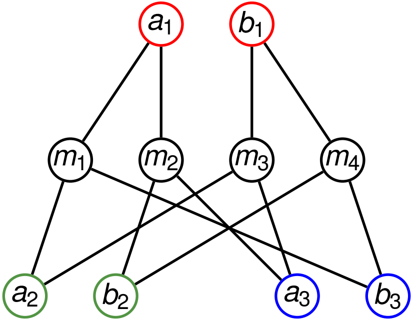
Thm. [CFI] No!

A simple graph property (now called the CFI property) checkable in $\mathrm{DTIME}[n]$, requires $v = \Omega(n)$ variables to express in C^v. Thus, CFI $\in$ **OIP** $- \mathrm{FPC}$

Proof of CFI Thm



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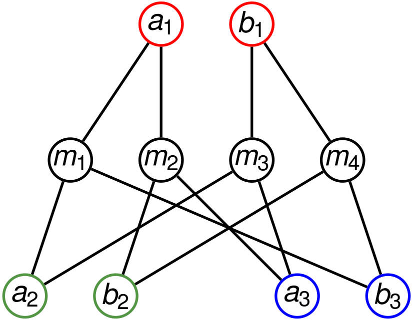

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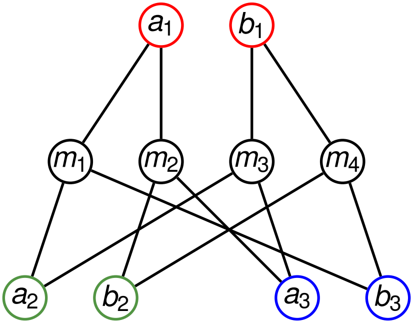

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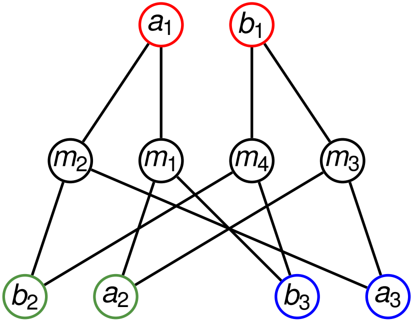

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Proof of CFI Thm

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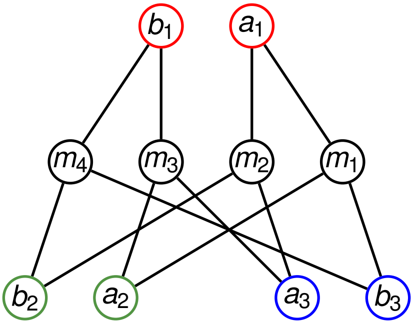

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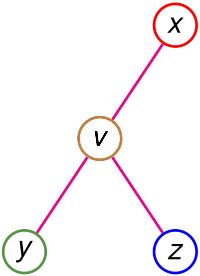
- Thus $X(G_n)$ has color class size 4.



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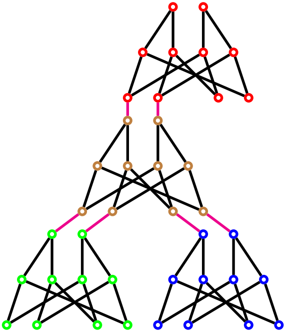

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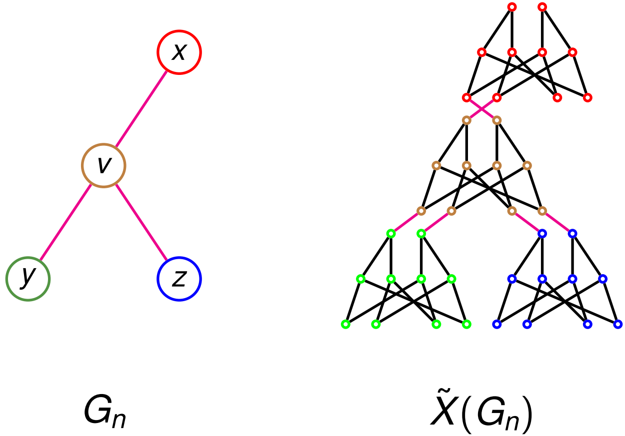

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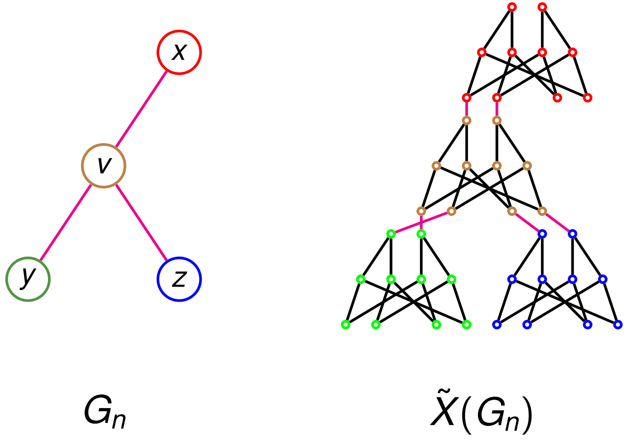

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Then $X'(G_n) \cong X(G_n)$ iff m is even and

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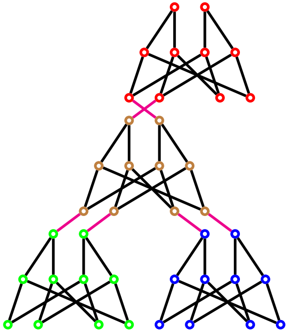

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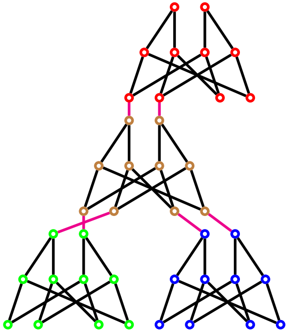

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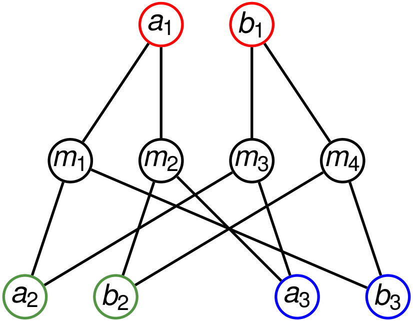

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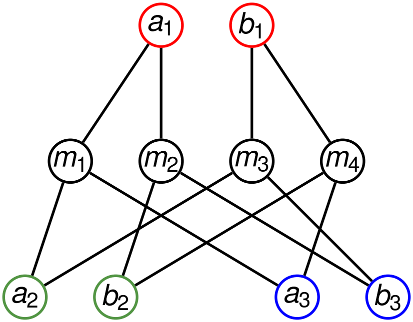

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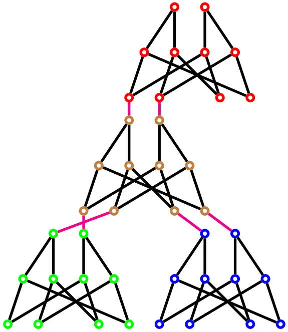
proof Use the ordering to label boundary pairs a_i, b_i when $a_i \le b_i$. Then count the number, m, of flips of vertices and edges mod 2. $X'(G) \in$ CFI iff m is even. $\square$

.



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Thus **Delilah** never loses. $\square$

Recap

We have shown that the linear-time CFI problem is in **OIP** $-\mathrm{FPC}$.

Cor. $\Omega(n)$ variables are needed to characterize graphs.

Recent Developments: FPC is Surprisingly Powerful

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Thm. [Anderson, Dawar and Holm] Linear Programming is in FPC.

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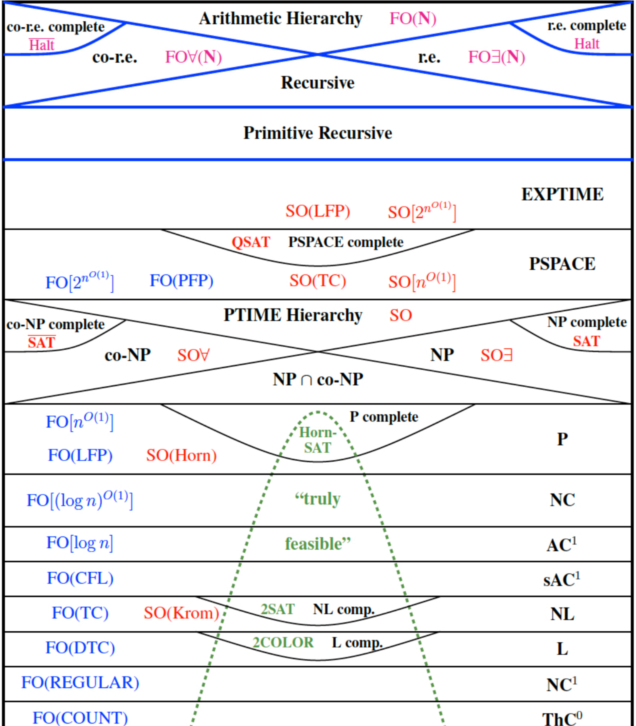
What I want: more natural extension to FPC that adds group theory and characterizes graphs using $O(\log n)$ variables.



Arithmetic Hierarchy FO(N)
co-r.e. complete
Halt
r.e. complete
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co-r.e. FO∀(N)
r.e. FO∃(N)
Recursive
Primitive Recursive
EXPTIME
SO(LFP) SO[2^{n^{O(1)}}]
QSAT PSPACE complete
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P
FO(LFP) SO(Horn)
FO[(log n)^{O(1)}]
"truly
NC
FO[log n]
feasible"
AC^1
FO(CFL)
sAC^1
FO(TC) SO(Krom)
2SAT NL comp.
NL
FO(DTC)
2COLOR L comp.
L
FO(REGULAR)
NC^1
FO(COUNT)
ThC^0