# “Robust” Lower Bounds

*for Communication and Stream Computation*



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**Graham Cormode** AT&T Labs
**Andrew McGregor** UC San Diego

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$x_1 \ldots x_{10}$

$x_{11} \ldots x_{20}$

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- *Consider random partitions:*

  Define error probability over coin flips and random split.

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- *Goal:* Evaluate $f(x_1, \ldots, x_n)$ given sequential access:

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  Upper Bounds: e.g., stream of i.i.d. samples.

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- Random-partition-CC bounds give random-order bounds

# Results

- *t-party Set-Disjointess:* Any protocol for $\Omega(t^2)$-player random-partition requires $\Omega(n/t)$ bits communication.

  ∴ 2-approx. for $k^{th}$ freq. moments requires $\Omega(n^{1-3/k})$ space.

- *Median:* Any $p$-round protocol for $p$-player random-partition requires $\Omega(m^{f(p)})$ where $f(p)=1/3^p$

  ∴ Polylog(m)-space algorithm requires $\Omega(\log\log m)$ passes.

- *Gap-Hamming:* Any one-way protocol for 2-player random-partition requires $\Omega(n)$ bits communicated.

  ∴ $(1+\varepsilon)$-approx. for $F_0$ or entropy requires $\Omega(\varepsilon^{-2})$ space.

- *Index:* Any one-way protocol for 2-player random-partition (with duplicates) requires $\Omega(n)$ bits communicated.

  ∴ Connectivity of a graph $G=(V, E)$ requires $\Omega(|V|)$ space.

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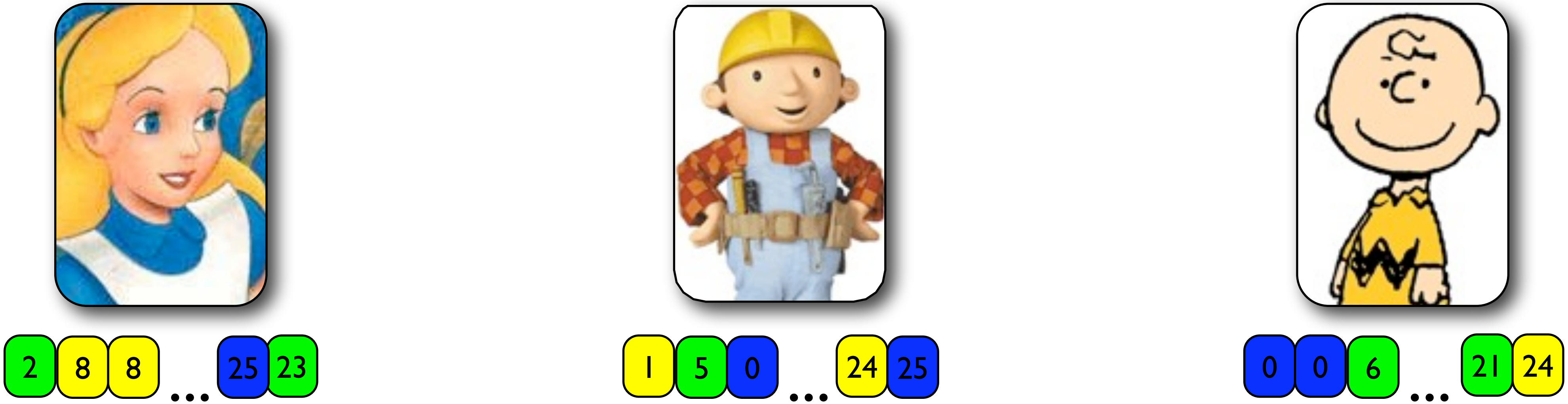


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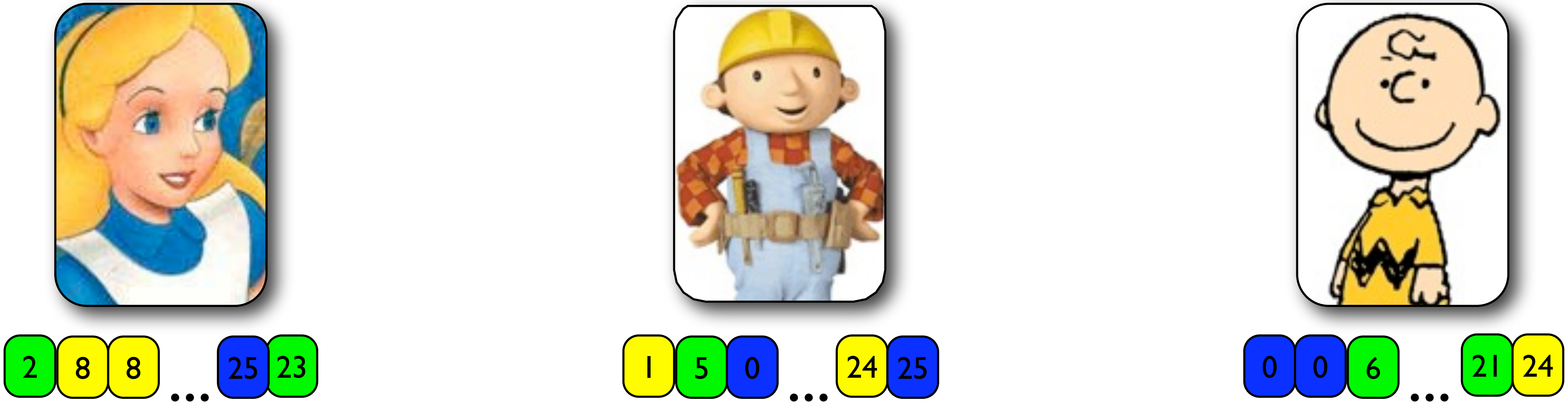


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- *Consider random input and public coins:*

  Issue #1: Need independence of input and partition.

  Issue #2: Generalize information statistics techniques.

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# Multi-Party Set-Disjointness

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- *Thm:* $\Omega(n/t)$ bound for random partition for $\Omega(t^2)$ players.

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- *Necessary Generalization:*
  Step 1: Condition "icost" on public coins.
  Step 2: Error of Π' is best $\delta$+Birthday$(t,p)$ error protocol.
  Step 3: Generalize result for public-coin protocols.

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- *Open Problem:* $\Omega(n^{1-2/k})$ bound for random order?

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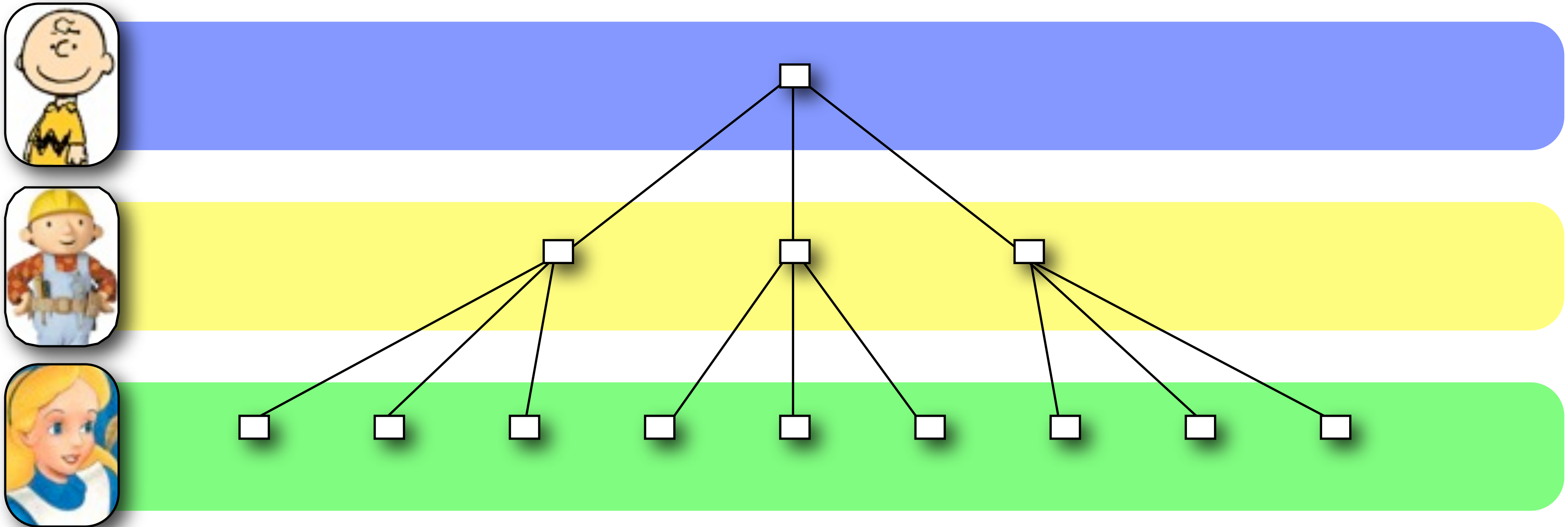
- *Our result:* Using random-partition-CC techniques we get simpler and tighter pass/space trade-offs...

# Tree Pointer Jumping (TPJ)...



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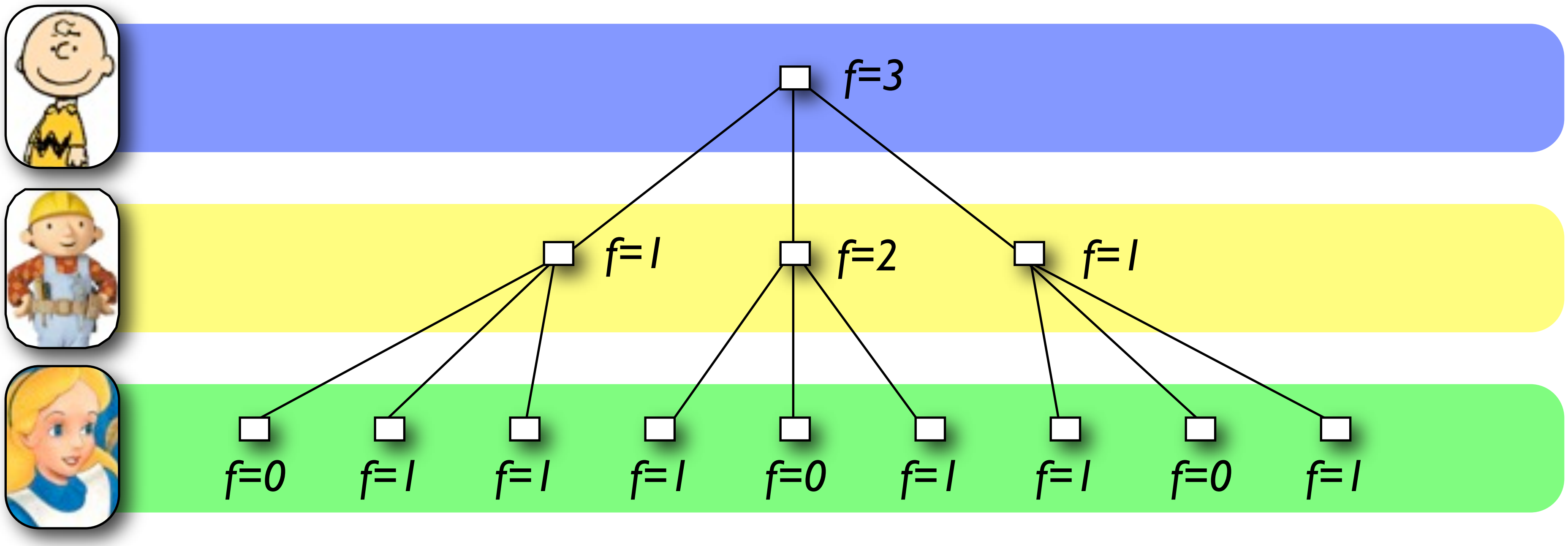


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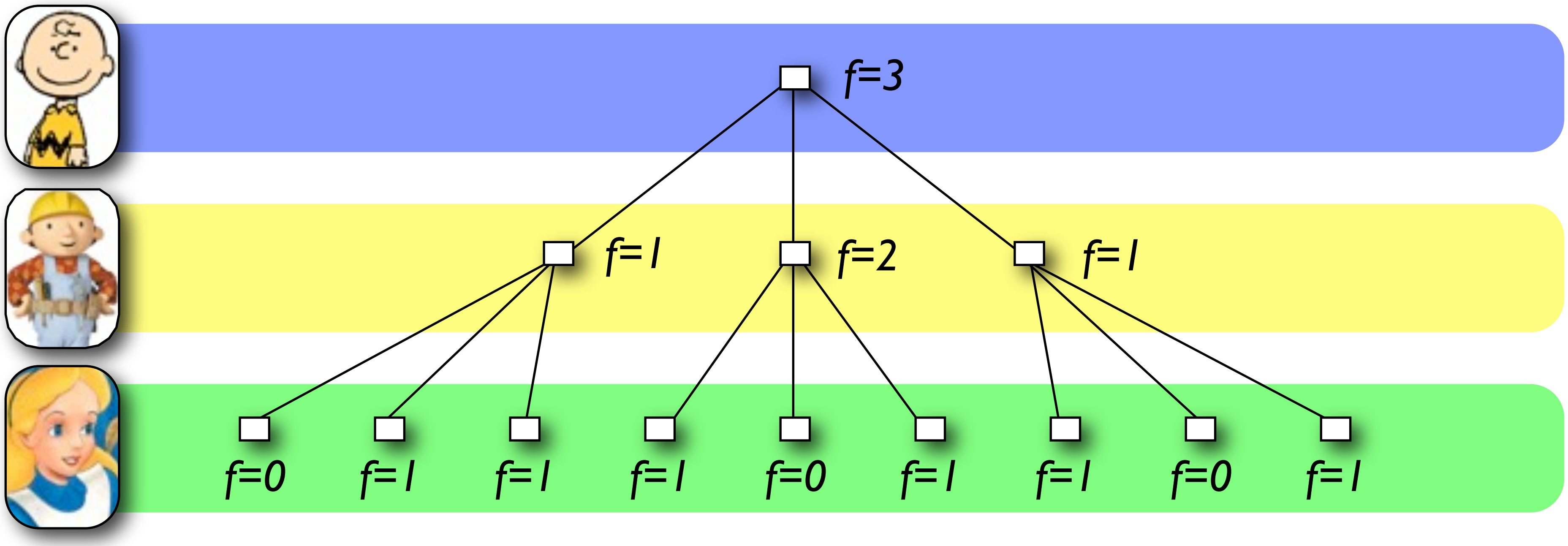


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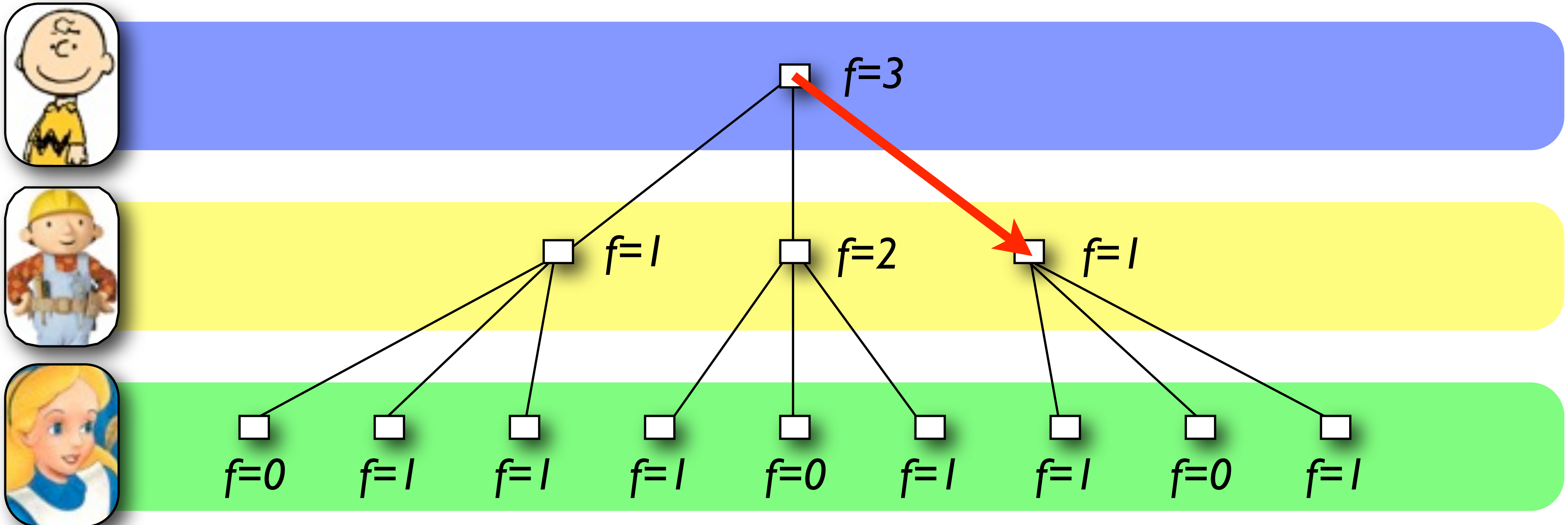


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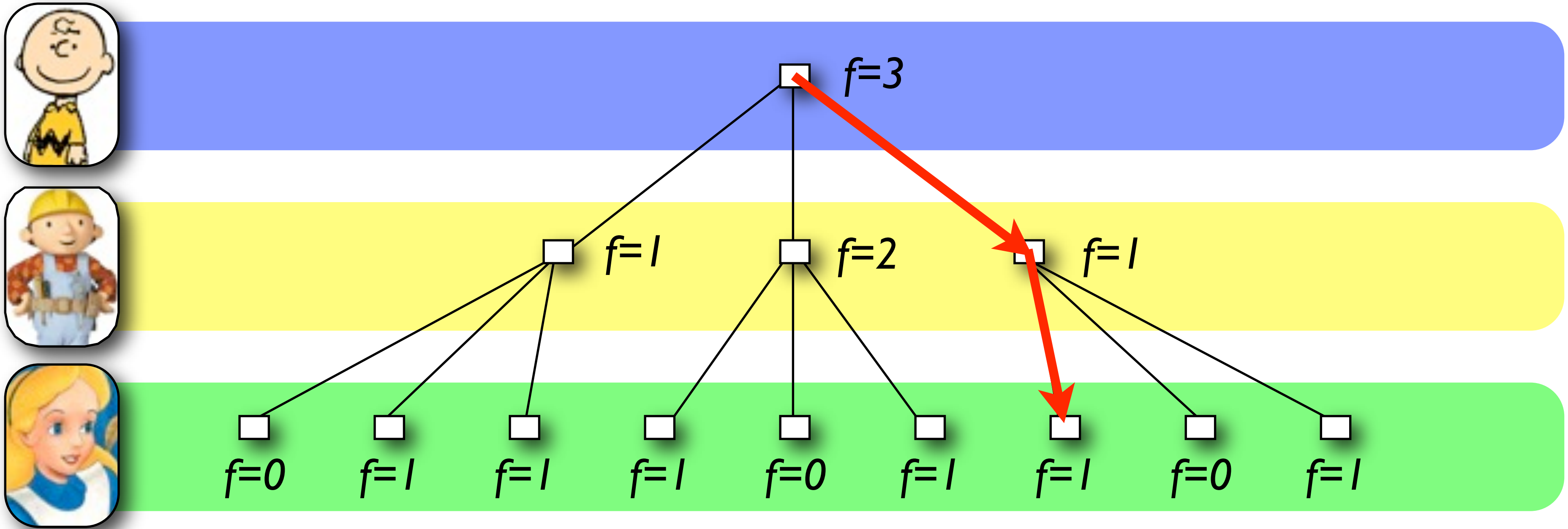


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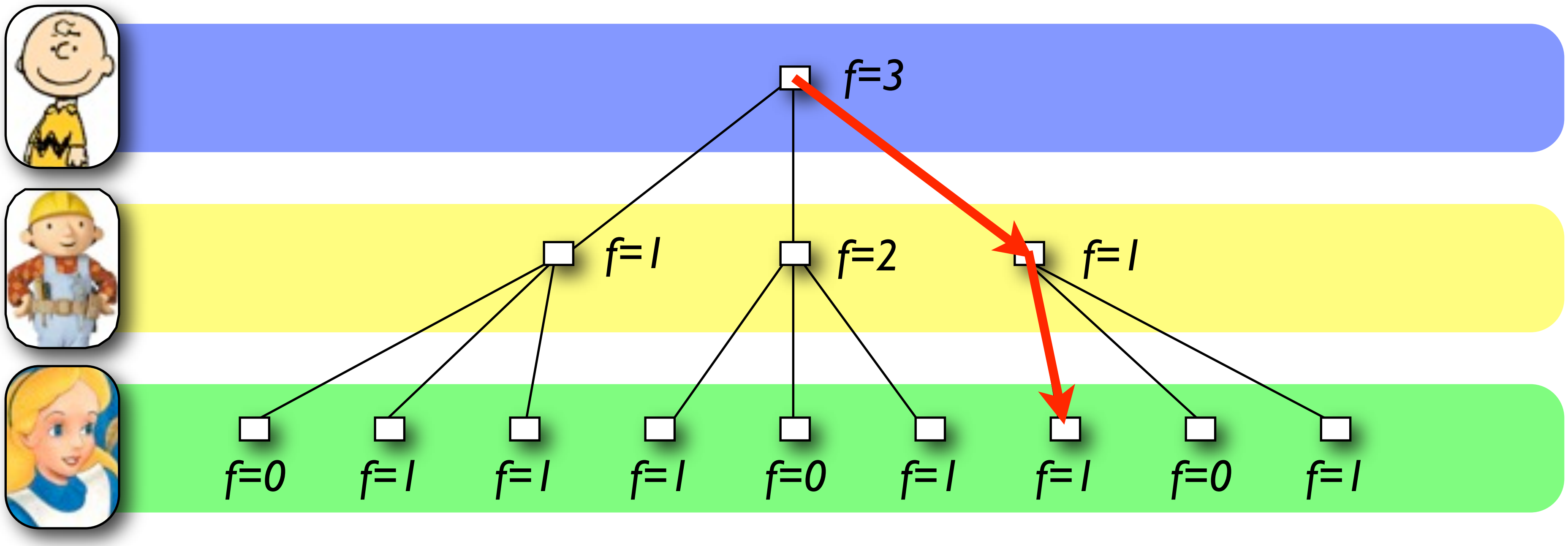


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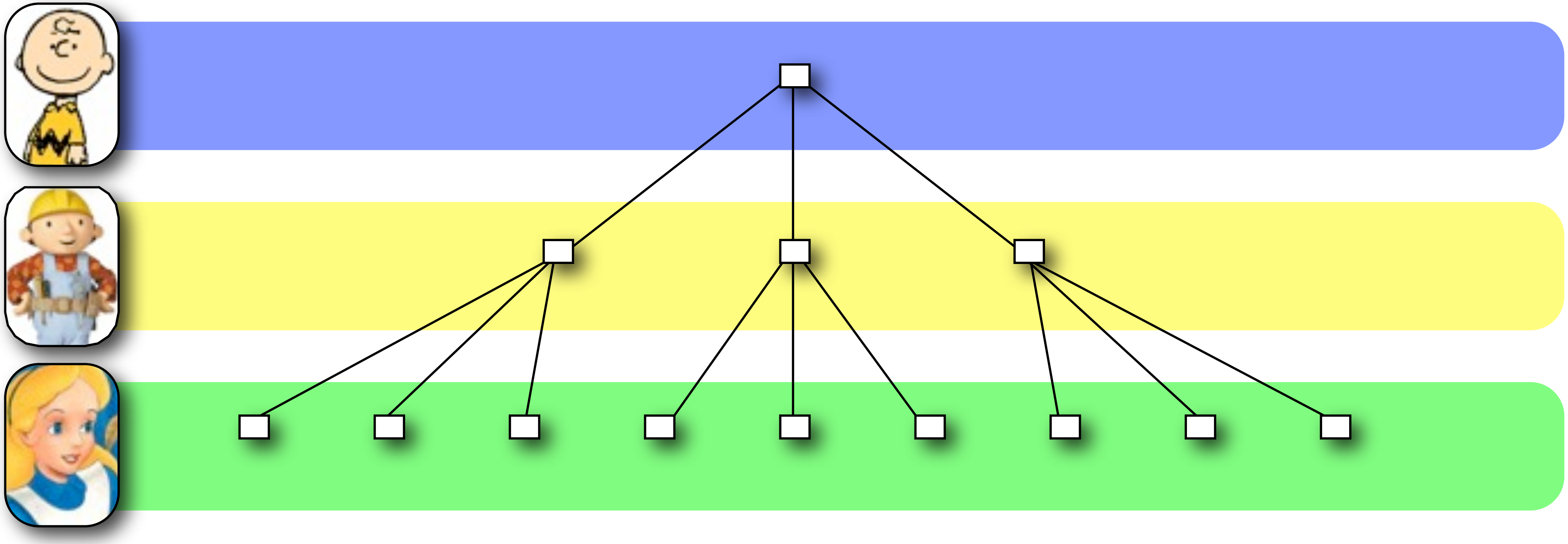

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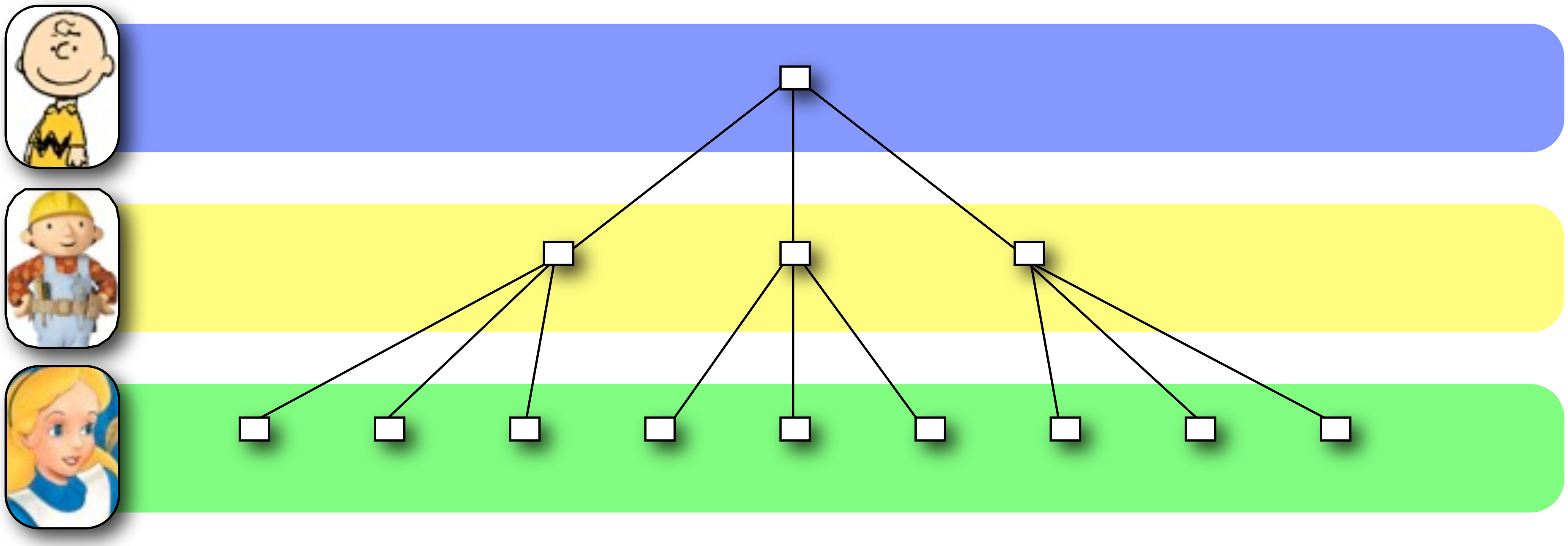

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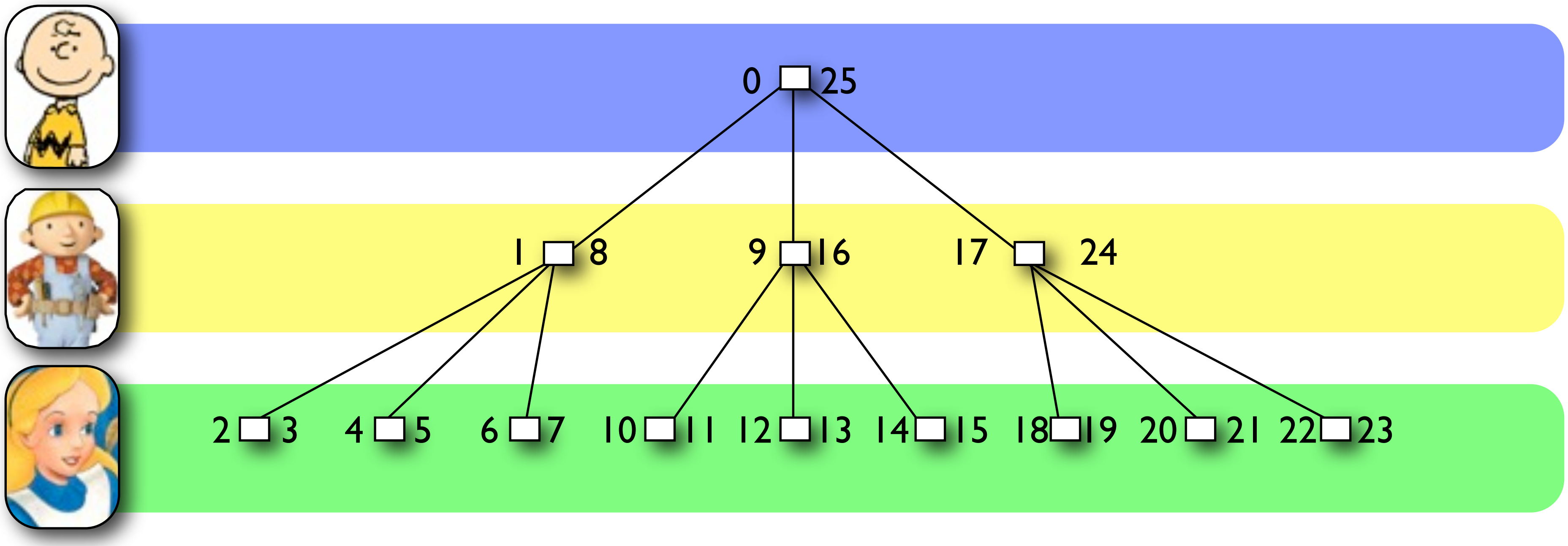


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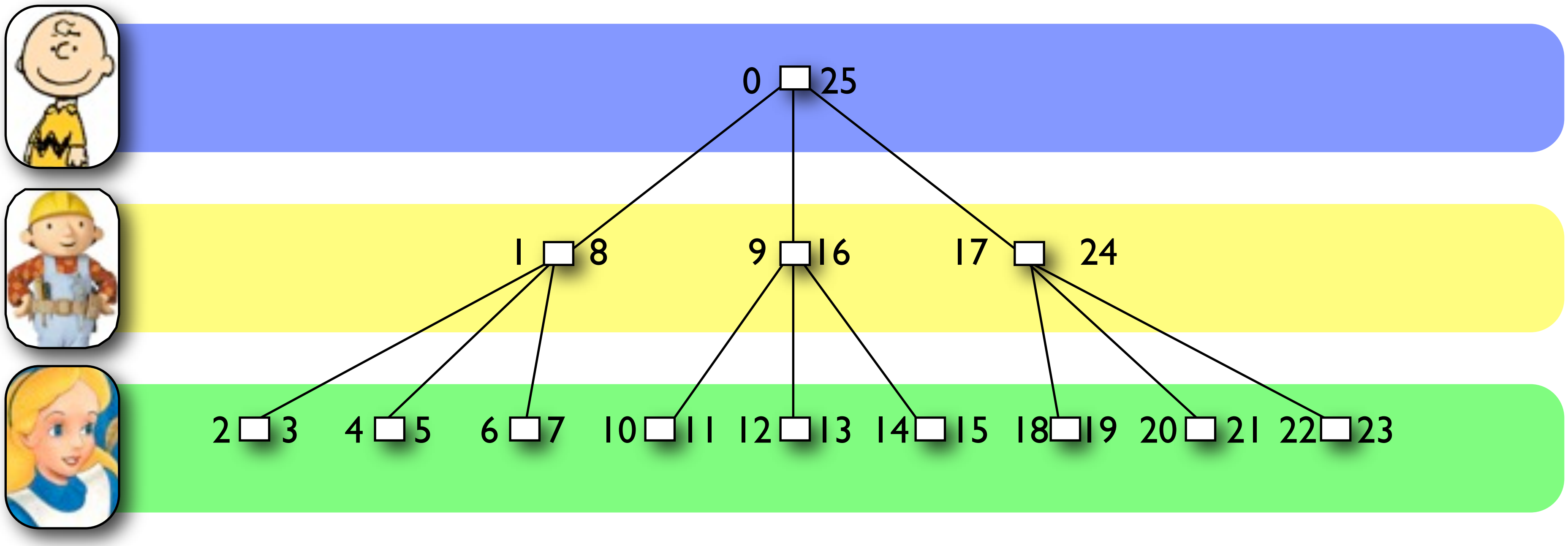


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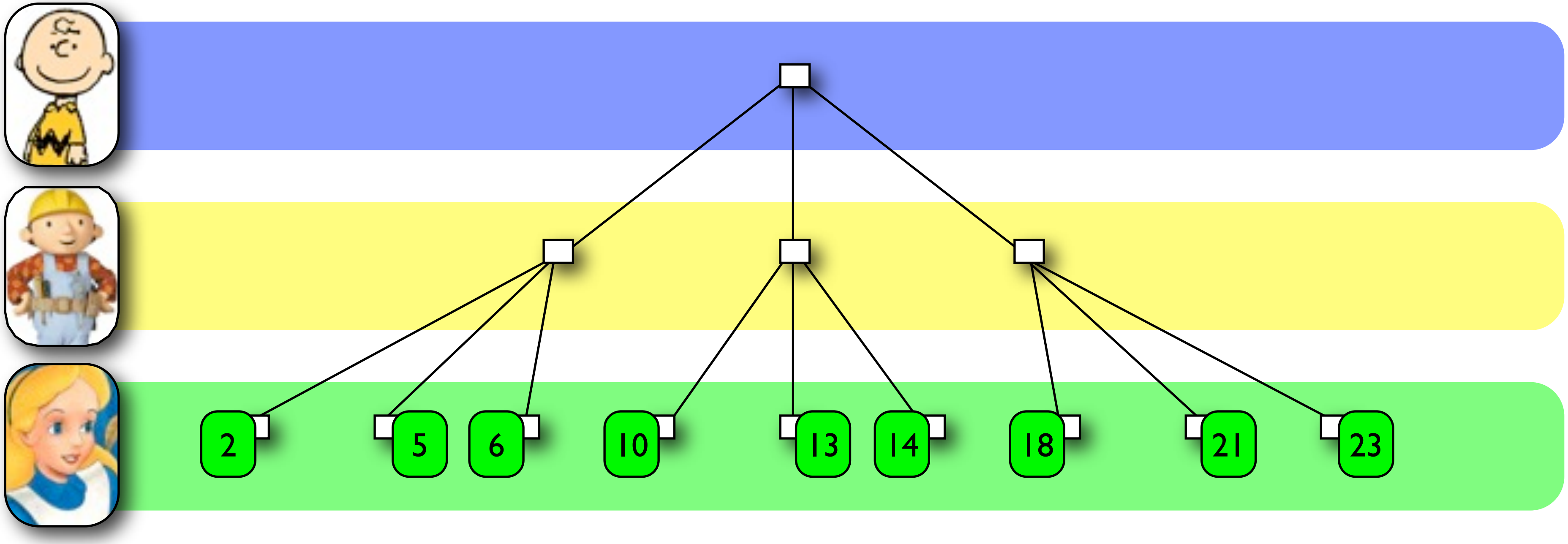


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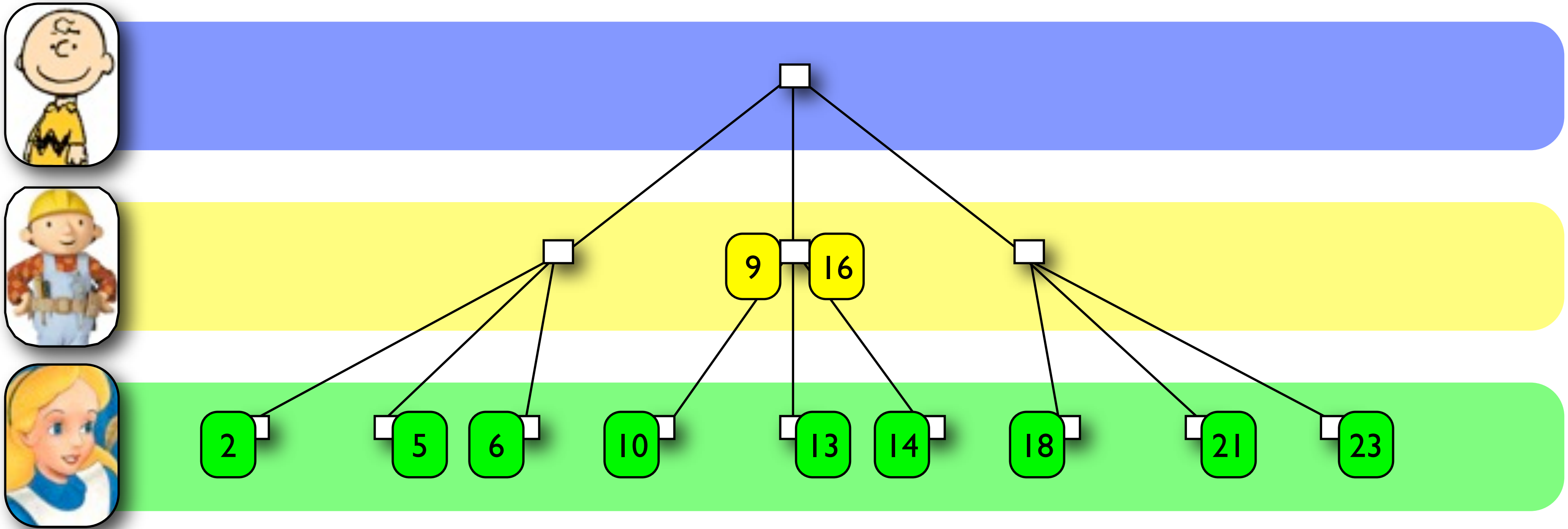


- With each node v associate two values α(*v*) < β(*v*) such that α(*v*) < α(*u*) < β(*u*) < β(*v*) for any descendent *u* of *v*.
- *For each node:* Generate multiple copies of α(*v*) and β(*v*) such that median of values corresponds to TPJ solution.

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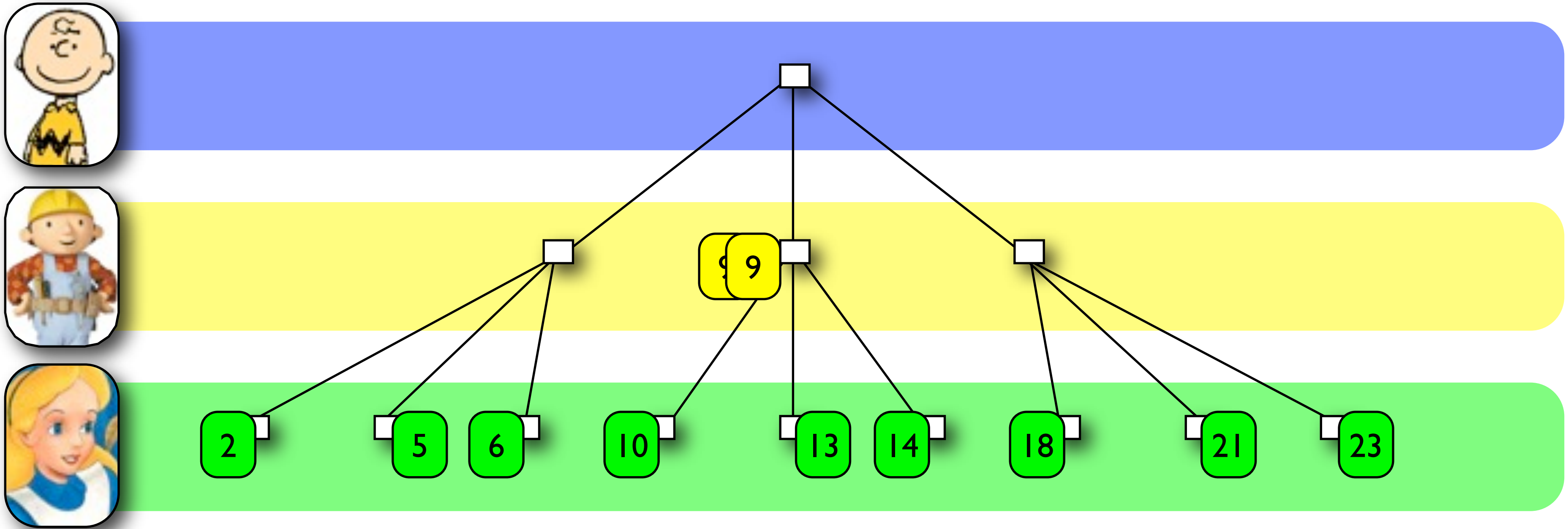


- With each node v associate two values $\alpha(v) < \beta(v)$ such that $\alpha(v) < \alpha(u) < \beta(u) < \beta(v)$ for any descendent *u* of *v*.

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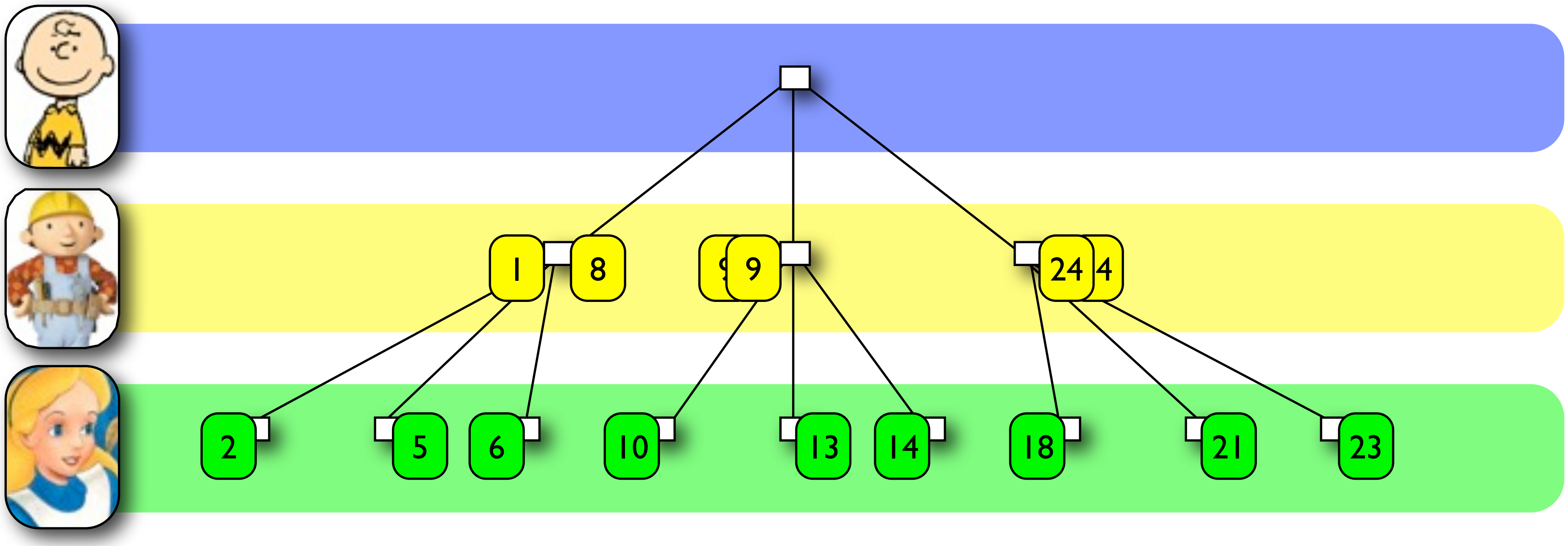


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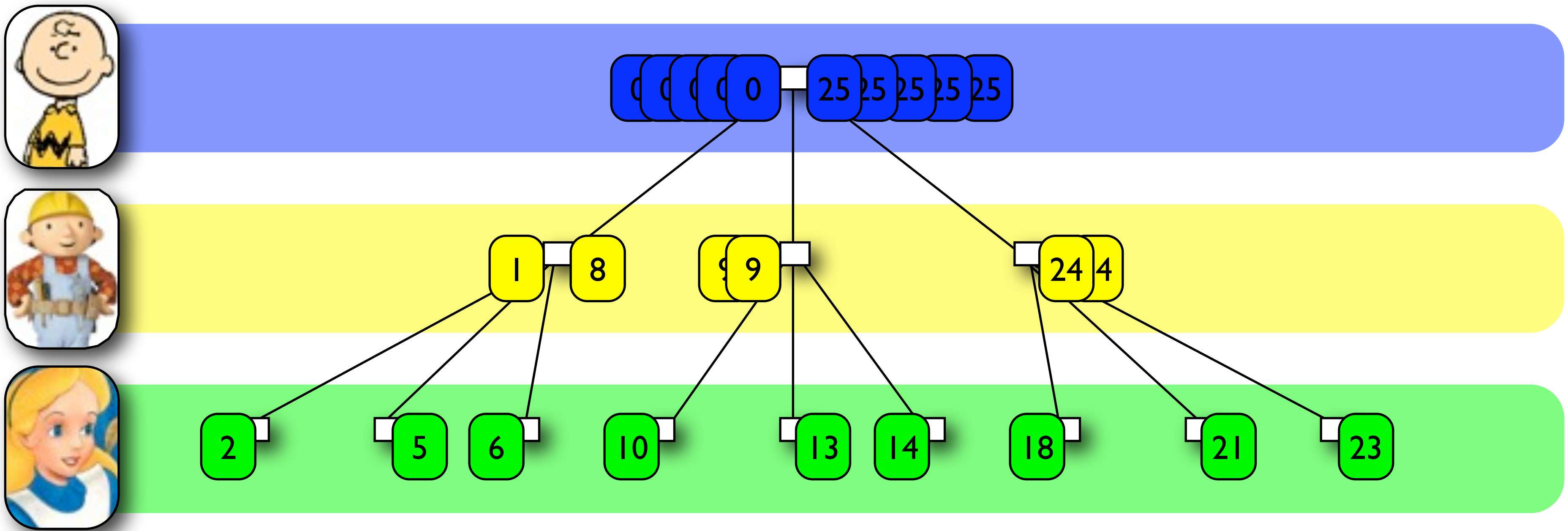


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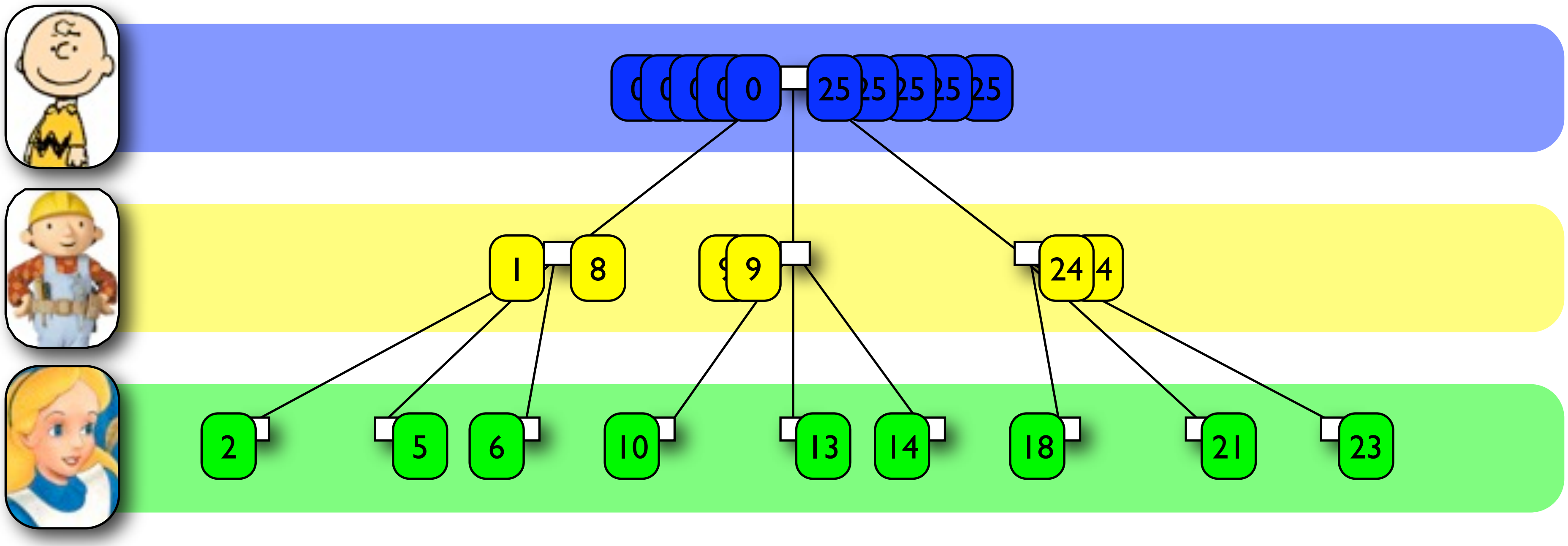


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- Relationship between *t* and # copies determines bound.

# *Simulating Random-Partition Protocol...*

  

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α α α β β β β β

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  1) Using public coin, players determine partition of tokens and set half to α and half to β.

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β α β    α β β    α β

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- *Thm:* Partition looks random if total number of tokens is greater than (max bias)$^2$. Hence, $m = \exp(2^p \lg t)$.

# *Summary*

Introduced notion of Robust Lower Bounds

Tight communication bounds for *disjointness*, *indexing*, *gap-hamming*, and improved *selection* bound.

Data streams bounds including *frequency moments*, *connectivity*, *entropy*, $F_0$, *quantile estimation*, ...

Many open problems... *Thanks!*