# CMPSCI 311: Introduction to Algorithms

Lecture 15: Dynamic Programming 4

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## Today

- All pairs shortest paths
- Dynamic programming failure
- Dynamic programming takeaways
- Planning and Decision Processes

## All-pairs shortest paths

- How fast can we compute all shortest paths in a graph?
  - Djikstra's gives $O(nm \log_2 n)$. (Requires non-negative weights)
  - Bellman-Ford gives $O(n^2 m)$. (Allows negative weights)
  - (new) Floyd-Warshall gives $O(n^3)$.

**Problem.** Given $G = (V, E, c)$ with non-negative weights, compute $n \times n$ array $M$ where $M[s,t]$ is the cost of shortest $s \rightsquigarrow t$ path.

- What are good subproblems?

## Floyd-Warshall algorithm

- Let $\text{cost}(s,t,k)$ be cost of shortest $s \rightsquigarrow t$ path using only vertices $\{1, \ldots, k\}$ as intermediate points.
- Consider $\text{cost}(s,t,n)$ for fixed $s, t$.
  - If $n$ not on shortest path, then $\text{cost}(s,t,n) = \text{cost}(s,t,n-1)$.
  - Otherwise, $\text{cost}(s,t,n) = \text{cost}(s,n,n-1) + \text{cost}(n,t,n-1)$.

$$\text{cost}(s,t,k+1) = \min \begin{cases} \text{cost}(s,t,k) \\ \text{cost}(s,k+1,k) + \text{cost}(k+1,t,k) \end{cases}$$

- Running time. $O(n^3)$.
- Recovering paths requires careful book-keeping.

## Interval Scheduling

**Problem.** Given $n$ shows with start time $s_i$ and finish time $f_i$, watch as many shows as possible, with no overlap.

- Greedy: order by $f_i$ (ascending), take next show if no conflict.
- Dynamic program:
  - Order by finish time $f_1 \leq f_2 \leq \ldots \leq f_n$
  - Compute $p(i) = \max\{j : f_j \leq s_i\}$.
  - $\text{VAL}(n) = \max\{\text{VAL}(p(n)) + 1, \text{VAL}(n-1)\}$.

## Another attempt

- Order shows arbitrarily, let $Q(i)$ be the shows that conflict with $i$ (including $i$).
- Consider optimal solution $O$,
  - If $n \notin O$ then $O$ is optimal on $\{1, \ldots, n-1\}$.
  - If $n \in O$ then $O$ is optimal on $\{1, \ldots, n-1\} \setminus Q(n)$.
- Generally, for set of shows $S$, if $i \in S$,

$$\text{VAL}(S) = \max\{\text{VAL}(S \setminus \{i\}), 1 + \text{VAL}(S \setminus Q(i))\}.$$

- How many subproblems? $\Omega(2^{n/2})$!

## Proof Idea

Suppose shows are $1, \ldots, n$ and show $i$ conflicts with $n - i + 1$.

- Process $\{1, \ldots, n\}$ requires $\{2, \ldots, n-1\}$ and $\{1, \ldots, n-1\}$.
- $\{2, \ldots, n-1\}$ requires $\{2, \ldots, n-2\}$ and $\{3, \ldots, n-2\}$.
- $\{1, \ldots, n-1\}$ requires $\{1, \ldots, n-2\}$ and $\{1, 3, \ldots, n-2\}$.
- Creates 4 distinct subproblems.

## Proof

- Suppose shows are $1, \ldots, n$ and show $i$ conflicts with $n - i + 1$.
- Represent subsets as binary strings of length $n$.
- Only worry about first $n/2$ bits (shows $1, \ldots, n/2$).
- Create binary tree, where at level $i$ process show $n - i + 1$.
  - Two subproblems, $i$th bit on and $i$th bit off.
- Generates all strings on $n/2$ bits $\Rightarrow \Omega(2^{n/2})$ subproblems.

## Dynamic Programming Takeaways

**Recipe**

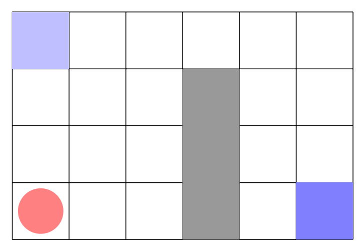
- Devise recursive form for solution
- Observe that recursive implementation involves redundant computation. (Often exponential time)
- Design iterative algorithm that solves all subproblems without redundancy.

**Concerns**

- What are the subproblems? How many are there?
  - Runtime and space complexity.

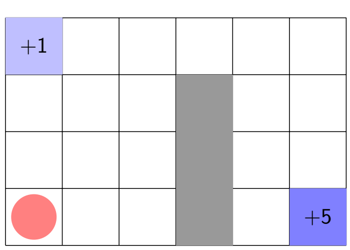
## Decision Processes

- Model of an agent performing a task in an environment.
- Used in AI, robotics, and many other places.



## Decision Process

- Set of states $S = \{1, \ldots, n\}$.
- Set of actions $A = \{1, \ldots, k\}$.
- Transition model: $T : S \times A \to S$.
- Reward function: $R : S \times A \to \mathbb{Z}$.
- Timer $H$.




## Trajectories

- Agent starts in $s_1$, takes action $a_1$, receives reward $R(s_1, a_1)$ and transitions to $s_2$, etc.
- Generates trajectory $s_1, a_1, r_1, s_2, a_2, r_2, \ldots, s_H, a_H, r_H$, where $r_h = R(s_h, a_h)$.
- Total reward is,

$$\sum_{h=1}^{H} r_h = \sum_{h=1}^{H} R(s_h, a_h)$$

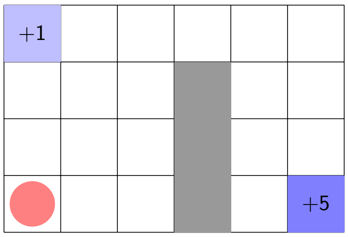
**Goal.** Choose actions to maximize total reward.

## Decision Process

- A policy chooses an action at every state and time,

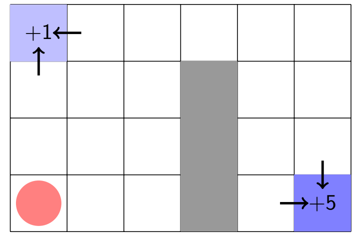
$$\pi : (S \times \{1, \ldots, H\}) \to A$$

**Goal.** Compute *policy* to maximize total reward.




## Example

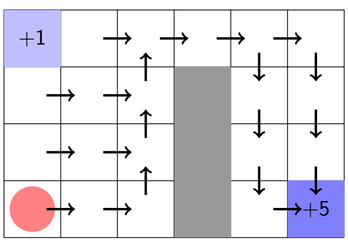
If $H = 1$:




$\pi^\star(\cdot, 1)$

## The Planning Problem

**Problem.** Compute optimal policy in decision process $(S, A, T, R, H)$.




$\pi^\star(\cdot, 11)$

## Base case

Consider $H = 1$.

- The optimal policy is,

$$\pi^\star(s, 1) = \operatorname*{argmax}_{a \in A} R(s, a)$$

- The optimal *values* are,

$$V^\star(s, 1) = \max_{a \in A} R(s, a)$$

- $V^\star(s, H)$ is maximum total reward you can achieve starting in state $s$ with $H$ actions.

## Inductive step

Consider arbitrary $h$.

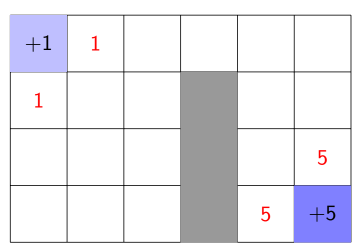
- If in state $s$, action $a$, receive $R(s, a)$ and transition to $T(s, a)$ with one less time point.
- How much more reward can you receive from $s' = T(s, a)$ with $h - 1$ actions left?

$$V^\star(s, h) = \max_{a \in A} \underbrace{R(s, a) + V^\star(T(s, a), h - 1)}_{Q^\star(s,a,h)}$$

- Policy is,

$$\begin{aligned}\pi^\star(s, h) &= \operatorname*{argmax}_{a \in A} R(s, a) + V^\star(T(s, a), h - 1) \\ &= \operatorname*{argmax}_{a \in A} Q^\star(s, a, h)\end{aligned}$$

## Example




$V^\star(\cdot, 1)$

## Example

| | | | | | |
|---|---|---|---|---|---|
| +1 | 1 | 1 | | | |
| 1 | 1 | | | | 5 |
| 1 | | | | 5 | 5 |
| | | | | 5 | +5 |

$V^{\star}(\cdot, 2)$

## Example

| | | | | | |
|---|---|---|---|---|---|
| +1 | 1 | 1 | 1 | | 5 |
| 1 | 1 | 1 | | 5 | 5 |
| 1 | 1 | | | 5 | 5 |
| 1 | | | | 5 | +5 |

$V^{\star}(\cdot, 3)$

## Example

| | | | | | |
|---|---|---|---|---|---|
| +1 | 1 | 1 | 1 | **5** | 5 |
| 1 | 1 | 1 | | 5 | 5 |
| 1 | 1 | 1 | | 5 | 5 |
| 1 | 1 | | | 5 | +5 |

$V^{\star}(\cdot, 4)$

## Value iteration

`ValueIteration`(T,R,H)
Initialize $V^{\star}(s, 0) = 0$ for all $s$.
Initialize $\pi^{\star}(s, h) = \texttt{null}$ for all $s, h$.
**for** $h = 1, \ldots, H$ **do**
  **for** each state $s$ **do**
    $V^{\star}(s, h) \leftarrow \max_a R(s, a) + V^{\star}(T(s, a), h - 1)$.
    $\pi^{\star}(s, h) \leftarrow \operatorname{argmax}_a R(s, a) + V^{\star}(T(s, a), h - 1)$.
  **end for**
**end for**
**Return** $\pi^{\star}$.

## Extensions

- Works without timer (under some conditions)
- Also works for stochastic (Markov) Decision Processes
- Reinforcement learning: Compute optimal policy when you don't know $T, R$
  - But can sample through experience.