CMPSCI 311: Introduction to Algorithms

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Announcements

- Homework 1 released (Due 2/7 11:59pm)
- Quiz 1 out (Due 1/30 11:59pm)
- Discussion section Friday

Plan

- Review: Asymptotics
  - $O(\cdot), \Omega(\cdot), \Theta(\cdot)$
  - Running time analysis
- Graphs
  - Motivation and definitions
  - Graph traversal

Review: Asymptotics

Definition $f(n) = O(g(n))$ if there exists n_0, c such that for all $n \geq n_0$, $f(n) \leq cg(n)$.

- g is an asymptotic upper bound on f.

Definition $f(n) = \Omega(g(n))$ if $g(n) = O(f(n))$.

- g is an asymptotic lower bound on f.

Definition $f(n) = \Theta(g(n))$ if $f(n) = O(g(n))$ and $g(n) = O(f(n))$.

- g is an asymptotically tight bound on f.

Algorithm design

- Formulate the problem precisely
- Design an algorithm to solve the problem
- Prove the algorithm is correct
- Analyze the algorithm's running time

Running Time Analysis

Mathematical analysis of worst-case running time of an algorithm as function of input size. Why these choices?

- Mathematical: describes the *algorithm*. Avoids hard-to-control experimental factors (CPU, programming language, quality of implementation), while still being predictive.
- Worst-case: just works. ("average case" appealing, but hard to analyze)
- Function of input size: allows predictions. What will happen on a new input?

Running time analysis

Mathematical analysis of worst-case running time of an algorithm as function of input size.

- To prove $O(f(n))$: Argue that for all n and *for all* inputs of size n the number of primitive operations is $O(f(n))$.
- To prove $\Omega(g(n))$: Argue that for all n, *there exists* some input of size n where the number of primitive operations is $\Omega(g(n))$.

Polynomial Time

Working definition of efficient

Definition: an algorithm runs in polynomial time if the number of primitive execution steps is at most cn^d, where n is the input size and c and d are constants.

- Matches practice: almost all practically efficient algorithms have this property
- Usually distinguishes a clever algorithm from a "brute force" approach ($n^d = O(2^n)$ for all constant d).
- Refutable: gives us a way of saying an algorithm is not efficient, or that no efficient algorithm exists.

Plan

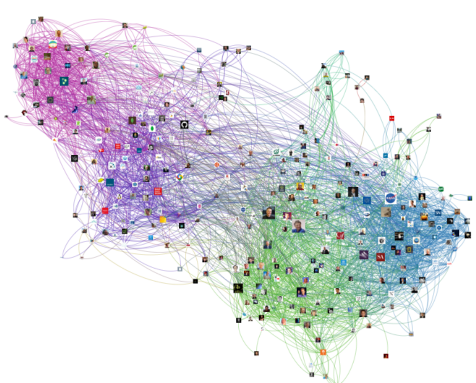
- Review: Asymptotics
  - $O(\cdot), \Omega(\cdot), \Theta(\cdot)$
  - Running time analysis
- Graphs
  - Motivation and definitions
  - Graph traversal

Questions

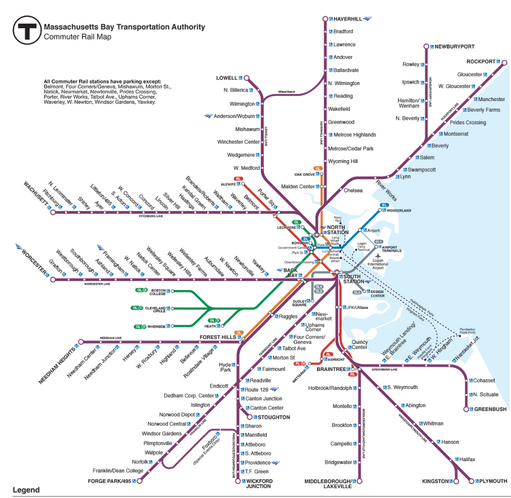
- Facebook: how many "degrees of separation" between me and Barack Obama?
- Google Maps: what is the shortest driving route from South Hadley to Florida?

Can we build algorithms to answer these questions?

Networks



Networks



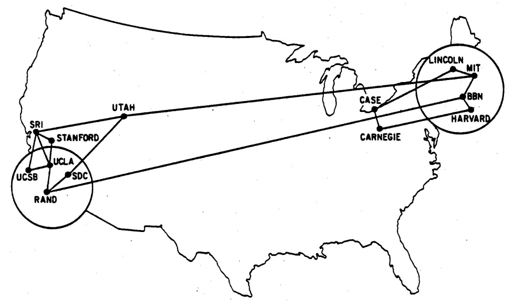
Graphs

A graph is a mathematical representation of a network

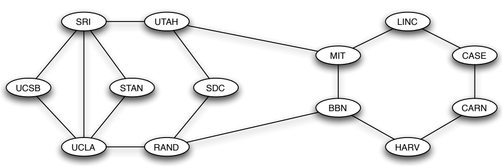
- Set of nodes (vertices) V
- Set of pairs of nodes (edges) E

Graph $G = (V, E)$

Example: Internet in 1970

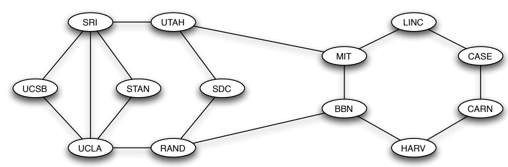


Example: Internet in 1970



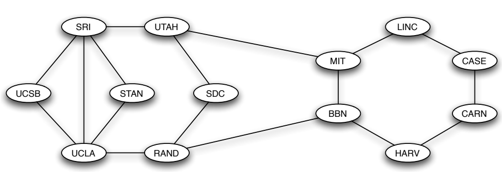
Definitions:
Vertex v, edge $e = (u, v)$, neighbor, incident, endpoints

Example: Internet in 1970



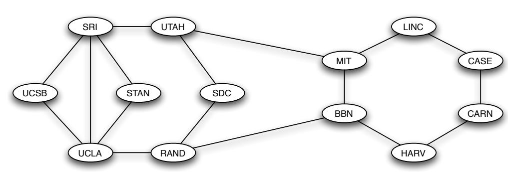
Definitions:
Path, cycle, path length, distance between two nodes

Example: Internet in 1970



Definitions:
Connected. Connected components.

Example: Internet in 1970



Definitions:
Tree = a connected undirected graph that does not contain a cycle
Rooted vs. unrooted trees

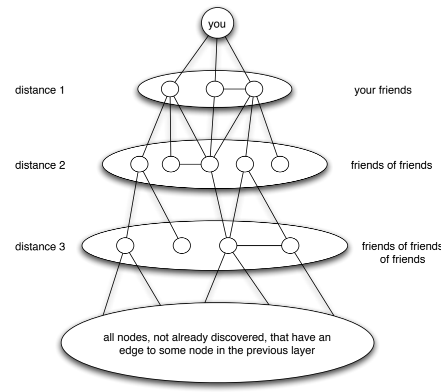
Graph Traversal

Thought experiment. World social graph. Is it connected? Is there a path between you and Barack Obama? How can you tell?

Answer: graph traversal! (BFS/DFS)

Breadth First Search

Traverse graph by exploring outward from starting node by distance. "Expanding wave"



Breadth-First Search: Layers

Define layer L_i = all nodes at distance exactly i from s.

Layers

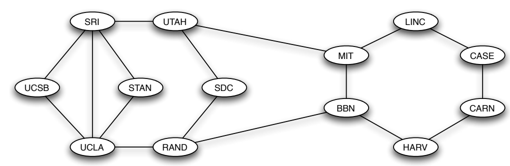
- $L_0 = \{s\}$
- $L_1 =$ all neighbors of L_0
- $L_2 =$ all nodes with an edge to L_1 that don't belong to L_0 or L_1
- ...
- $L_{i+1} =$ nodes with an edge to L_i that don't belong to any earlier layer.

$$L_{i+1} = \{v : \exists(u, v) \in E, u \in L_i, v \notin (L_0 \cup \ldots \cup L_i)\}$$

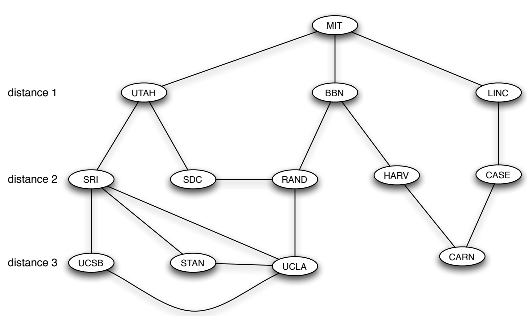
Observation: There is a path from s to t if and only if t appears in some layer.

BFS

Exercise: draw the BFS layers for a BFS starting from MIT

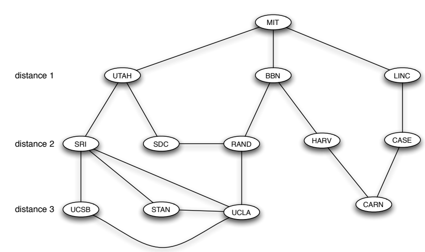


BFS Tree



We can use BFS to make a tree.

BFS Tree



Claim: let T be the tree discovered by BFS on graph $G = (V, E)$, and let (x, y) be any edge of G. Then the layer of x and y in T differ by at most 1.

Proof on board

BFS and non-tree edges

Claim: let T be the tree discovered by BFS on graph $G = (V, E)$, and let (x, y) be any edge of G. Then the layer of x and y in T differ by at most 1.
Proof

- Suppose $x \in L_i$ and $y \in L_j$ with $i < j - 1$ but edge (x, y) exists.
- When BFS visits x, either y is already discovered or not.
  - If y is already discovered, then $j \leq i$. Contradiction.
  - Otherwise since $(x, y) \in E$, y is added to L_{i+1}. Contradiction.

A More General Strategy

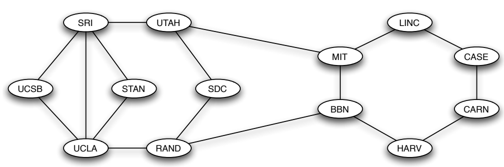
To explore the connected component, add any node v for which

- (u, v) is an edge
- u is explored, but v is not

Picture on board

DFS

Depth-first search: keep exploring from the most recently added node until you have to backtrack.
Example.



Recursive DFS

DFS(u)
  Mark u as "Explored"
  for each edge (u, v) incident to u **do**
    if v is not marked "Explored" **then**
      Recursively invoke DFS(v)
    end if
  end for

Example on board

DFS Tree

Claim: let T be a depth-first search tree for graph $G = (V, E)$, and let (x, y) be an edge that is in G but not T (a "non-tree edge"). Then either x is an ancestor of y or y is an ancestor of x in T.
proof on board

DFS and Non-tree edges

Claim: let T be a depth-first search tree for graph $G = (V, E)$, and let (x, y) be an edge that is in G but not T (a "non-tree edge"). Then either x is an ancestor of y or y is an ancestor of x in T.
Proof

- Suppose not and suppose that x is reached first by DFS.
- Before leaving x, we must examine (x, y).
- Since $(x, y) \notin T$, y must have been explored by this time.
- But y was not explored when we arrived at x by assumption.
- Thus y was explored during the execution of `DFS`(x).
- Implies x is ancestor of y.

Using Graph Traversal

Definition: the connected component $C(v)$ of node v is the set of all nodes with a path to v.

Easy claim: for any two nodes s and t either $C(s) = C(t)$, or $C(s)$ and $C(t)$ are disjoint.

Picture/example on board

Finding Connected Components

Traverse entire graph even if not connected.
Extract connected components.

```
while There is some unexplored node s do
    BFS(s)                          ▷ Run BFS starting from s.
    Extract connected component C(s).
end while
```

Running time?
What's the running time of BFS?

Summary So Far

- Graph – definitions
- Graph traversals – BFS, DFS, and some properties
- Finding connected components

- Next – Implementation and run-time analysis.

Representing a graph

Adjacency List Representation.

- Nodes numbered $1, \ldots, n$.
- $\texttt{Adj}[v]$ points to a list of all of v's neighbors.
- Example

Implementing BFS

Maintain set of explored nodes and discovered

- Explored = have seen this node and explored its outgoing edges
- Discovered = the "frontier". Have seen the node, but not explored its outgoing edges.

Picture on board

BFS Implementation

Let A = Queue of discovered nodes (FIFO)

```
Traverse(s)
  Put s in A
  while A is not empty do
      Take a node v from A
      if v is not marked "explored" then
          Mark v as "explored"
          for each edge (v, w) incident to v do
              Put w in A                    ▷ w is discovered
          end for
      end if
  end while
```

Note: one part of this algorithm seems really dumb. Why?
Can put multiple copies of a node in A. ("Rediscover it many times")

BFS Implementation

Let A = Queue of discovered nodes (FIFO)
Traverse(s)
  Put s in A
  while A is not empty **do**
    Take a node v from A
    if v is not marked "explored" **then**
      Mark v as "explored"
      for each edge (v, w) incident to v **do**
        Put w in A $\triangleright$ w is discovered
      end for
    end if
  end while

Is this BFS?

Summary

Definitions

- $G = (V, E), n = |V|, m = |E|$
- neighbor, incident, cycle, path, connected

BFS and DFS

- Two ways to traverse a graph, each produces a tree
- BFS tree: shallow and wide ("bushy")
- DFS tree: deep and narrow ("scraggly")
- Connected Components