Esimtating local maxima from discrete sample points

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Abstract

This document discusses the problem of trying to find the position of the exact maximum of a continuous function $f(x)$ when we are only given samples of the function at various discrete points. This is done by assuming the function is locally shaped like a parabola.

1 Subpixel maxima

Suppose we have a function of one variable, $f(x)$, and it has been sampled at several values of x:

$$
\begin{aligned}
&f(0) = 5\\
&f(1) = 7\\
&f(2) = 10\\
&f(3) = 8\\
&f(4) = 7\\
&...
\end{aligned}
$$

While $f(2)$ gives the highest local value, it is not clear where the highest value of the underlying continuous function $f(x)$ is. However, if we assume that $f(x)$ looks like a parabola (i.e., a quadratic) in the local vicinity of $x = 2$, then we can compute the local maximum of $f(x)$ near $x = 2$ in the following way.

First we make an expression for the Taylor series of $f(x)$ at $x = 2$:

$$f(x) = f(x)\Big|_{x=2} + \frac{df}{dx}\Big|_{x=2}(x-2) + \frac{1}{2}\frac{d^2f}{dx^2}\Big|_{x=2}(x-2)^2.$$

Assume for the moment that we have been given the values of the first and second derivatives at $x = 2$:

$$\frac{df}{dx}\Big|_{x=2} = 0.5,$$

and

$$\frac{d^2f}{dx^2}\Big|_{x=2} = -5.$$

In this case, we can rewrite our Taylor series as

$$f(x) = 10 + 0.5(x-2) - \frac{5}{2}(x-2)^2.$$

Differentiating, and setting equal to zero, we have

$$5(x-2) = 0.5,$$

$$5x = 10.5,$$

or

$$x = 2.1.$$

We can see that if the curve is locally a parabola, then the maximum will be at 2.1, which is just to the right of $x = 2$.

1.1 Derivative estimates

The only remaining question is, how do we obtain the first and second derivatives used above? There are many ways to estimate derivatives from sample points. In particular, the first derivative at $x = 2$ could be estimated as

$$f(2) - f(1),$$

$$f(3) - f(2),$$

or

$$
\begin{aligned}
& \frac{1}{2}[(f(3) - f(2)) + (f(2) - f(1))] && (1) \\
= \;& \frac{1}{2}[f(3) - f(1)]. && (2)
\end{aligned}
$$

Intuitively, the first two choices for computing the derivative seem suspect, since they are biased towards one side of the point or the other.

Note that for a general parabola, $f(x) = ax^2 + bx + c$, we have

$$\left.\frac{df}{dx}\right|_{x=2} = [2ax + b]\Big|_{x=2} = 4a + b.$$

Also note that

$$f(1) = a + b + c$$

and

$$f(3) = 9a + 3b + c.$$

Hence,

$$
\begin{aligned}
& \frac{1}{2}[f(3) - f(1)] \\
= \;& \frac{1}{2}[9a + 3b + c - (a + b + c)] \\
= \;& 4a + b.
\end{aligned}
$$

Thus, our formula for estimating the derivative at a given point (from equation (2)) gives us the exact value of the derivative when the function is a parabola!

For the second derivative, we will use the filter $[1 \; -2 \; 1]$ to compute the second derivative around a point.