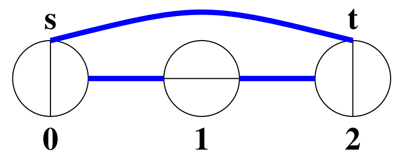
Tarski's Inductive Definition of Truth

$$
\begin{array}{lcll}
G & \models & t_1 = t_2 & \Leftrightarrow \quad t_1^G = t_2^G \\
G & \models & P(t_1, \ldots, t_r) & \Leftrightarrow \quad (t_1^G, \ldots, t_r^G) \in P^G \\
G & \models & \neg\varphi & \Leftrightarrow \quad G \not\models \varphi \\
G & \models & \varphi \wedge \psi & \Leftrightarrow \quad G \models \varphi \quad \textbf{and} \quad G \models \psi \\
G & \models & \varphi \vee \psi & \Leftrightarrow \quad G \models \varphi \quad \textbf{or} \quad G \models \psi \\
G & \models & \forall x(\varphi) & \Leftrightarrow \quad \textbf{for all } a \in U^G \quad (G, a/x) \models \varphi \\
G & \models & \exists x(\varphi) & \Leftrightarrow \quad \textbf{there exists an } a \in U^G \quad (G, a/x) \models \varphi
\end{array}
$$

Example 22.1 (25 examples of checking if $G \models \varphi$) For each of the following FO formulas, α_1, $1 \le i \le 5$, find the unique graph, $G_{g(i)}$ that satisfies it.

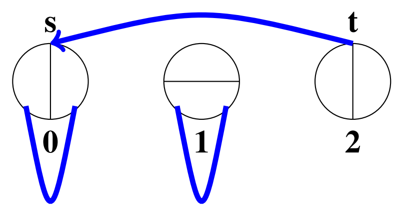
$$\alpha_1 \stackrel{\text{def}}{=} \forall xy\,(E(x,y) \rightarrow E(y,x))$$

G_1



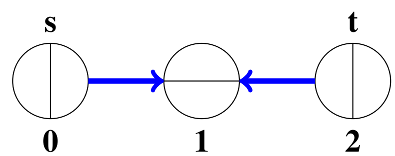
$$\alpha_2 \stackrel{\text{def}}{=} \exists x\,\forall y\,(E(x,y) \vee E(y,x) \rightarrow x = y)$$

G_2



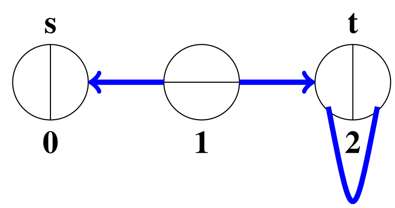
$$\alpha_3 \stackrel{\text{def}}{=} \begin{array}{l}\forall xy\,(E(x,y) \quad \rightarrow \\ \quad \neg(H(x) \wedge H(y)) \wedge \neg(V(x) \wedge V(y)))\end{array}$$

G_3



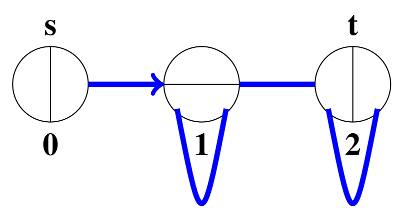
$$\alpha_4 \stackrel{\text{def}}{=} \exists x\,\forall y\,(H(x) \wedge \neg E(y,x))$$

G_4



$$\alpha_5 \stackrel{\text{def}}{=} \exists! x\,\exists! y\; E(x,y)$$

G_5

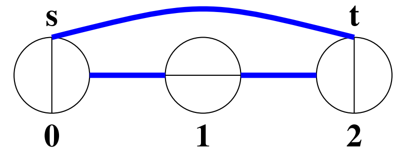


First convert the formulas to equivalent formulas in **NNF**:

$$
\begin{aligned}
\exists! x\, \exists! y\; E(x,y) \;&\equiv\; \exists x\, \forall x_1\, (\exists! y\, E(x,y) \;\wedge\; (\exists! y\, E(x_1,y) \;\rightarrow\; x_1 = x)) \\
&\equiv\; \exists x\, \forall x_1\, (\exists! y\, E(x,y) \;\wedge\; (\neg \exists! y\, E(x_1,y) \;\vee\; x_1 = x)) \\
&\equiv\; \exists x\, \forall x_1\, (\exists y\, \forall y_1\, (E(x,y) \;\wedge\; (\neg E(x,y_1) \vee y_1 = y)) \;\wedge \\
&\qquad \neg(\exists y_2\, \forall y_3\, (E(x_1,y_2) \wedge (\neg E(x_1,y_3) \vee y_3 = y_2))) \;\vee\; x_1 = x) \\
&\equiv\; \exists x\, \forall x_1\, \exists y\, \forall y_1\, ((E(x,y) \;\wedge\; (\neg E(x,y_1) \vee y_1 = y)) \;\wedge \\
&\qquad \forall y_2\, \exists y_3\, \neg(E(x_1,y_2) \wedge (\neg E(x_1,y_3) \vee y_3 = y_2))) \;\vee\; x_1 = x) \\
&\equiv\; \exists x\, \forall x_1\, \exists y\, \forall y_1\, \forall y_2\, \exists y_3\, ((E(x,y) \;\wedge\; (\neg E(x,y_1) \vee y_1 = y)) \;\wedge \\
&\qquad (\neg E(x_1,y_2) \vee \neg(\neg E(x_1,y_3) \vee y_3 = y_2))) \;\vee\; x_1 = x) \\
&\equiv\; \exists x\, \forall x_1\, \exists y\, \forall y_1\, \forall y_2\, \exists y_3\, ((E(x,y) \;\wedge\; (\neg E(x,y_1) \vee y_1 = y)) \;\wedge \\
&\qquad (\neg E(x_1,y_2) \vee (E(x_1,y_3) \wedge y_3 \neq y_2))) \;\vee\; x_1 = x)
\end{aligned}
$$

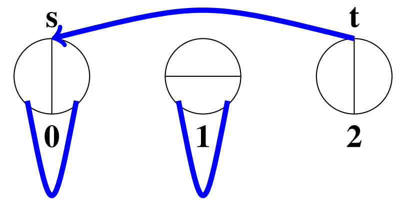
$$\alpha_1 \stackrel{\text{def}}{=} \forall xy\,(\neg E(x,y) \ \vee \ E(y,x))$$

G_1



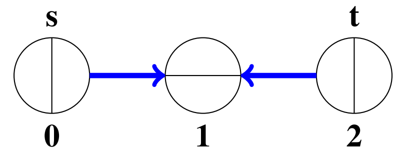
$$\alpha_2 \stackrel{\text{def}}{=} \exists x\,\forall y\,((\neg E(x,y) \wedge \neg E(y,x)) \ \vee \ x = y)$$

G_2



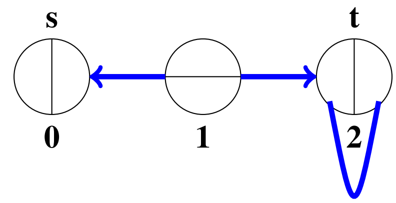
$$\alpha_3 \stackrel{\text{def}}{=} \begin{array}{l} \forall xy\,(\neg E(x,y) \quad \vee \\ \quad (\neg H(x) \vee \neg H(y)) \wedge (\neg V(x) \vee \neg V(y))) \end{array}$$

G_3



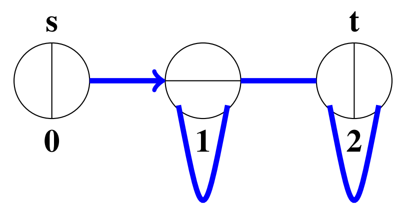
$$\alpha_4 \stackrel{\text{def}}{=} \exists x\,\forall y\,(H(x) \wedge \neg E(y,x))$$

G_4



$$\alpha_5 \stackrel{\text{def}}{=} \begin{array}{l} \exists x \forall x_1 \exists y \forall y_1 \forall y_2 \exists y_3\,((E(x,y) \wedge (\neg E(x,y_1) \vee y_1 = y)) \wedge \\ \quad (\neg E(x_1,y_2) \vee (E(x_1,y_3) \wedge y_3 \neq y_2))) \ \vee \ x_1 = x) \end{array}$$

G_5



Use Tarski's Definition of Truth to test whether $G_1 \models \alpha_1$.

$$G \models t_1 = t_2 \quad \Leftrightarrow \quad t_1^G = t_2^G$$

$$G \models P(t_1, \ldots, t_r) \quad \Leftrightarrow \quad (t_1^G, \ldots, t_r^G) \in P^G$$

$$G \models \neg\varphi \quad \Leftrightarrow \quad G \not\models \varphi$$

$$G \models \varphi \wedge \psi \quad \Leftrightarrow \quad G \models \varphi \ \textbf{ and } \ G \models \psi$$

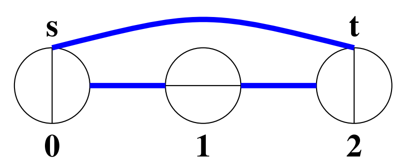
$$G \models \varphi \vee \psi \quad \Leftrightarrow \quad G \models \varphi \ \textbf{ or } \ G \models \psi$$

$$G \models \forall x(\varphi) \quad \Leftrightarrow \quad \textbf{for all } a \in U^G \quad (G, a/x) \models \varphi$$

$$G \models \exists x(\varphi) \quad \Leftrightarrow \quad \textbf{there exists an } a \in U^G \quad (G, a/x) \models \varphi$$

$$G_1 \models \forall xy\,(\neg E(x,y) \ \vee \ E(y,x))$$

G_1



$$\text{iff} \quad \bigwedge_{a=0}^{2} G_1, a/x \models \forall y\,(\neg E(x,y) \ \vee \ E(y,x))$$

$$\text{iff} \quad \bigwedge_{a=0}^{2} \bigwedge_{b=0}^{2} G_1, a/x, b/y \models \neg E(x,y) \ \vee \ E(y,x)$$

9 facts to check; all true:

$G_1,\ 0/x,\ 0/y \models \neg E(x,y) \ \vee \ E(y,x)$ because $G_1,\ 0/x,\ 0/y \models \neg E(x,y)$

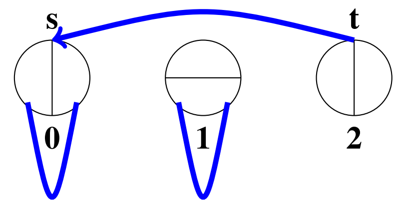
$G_1,\ 0/x,\ 1/y \models \neg E(x,y) \ \vee \ E(y,x)$ because $G_1,\ 0/x,\ 1/y \models E(y,x)$

$\vdots \quad \models \neg E(x,y) \ \vee \ E(y,x)$ because $\vdots$

$G_1,\ 2/x,\ 2/y \models \neg E(x,y) \ \vee \ E(y,x)$ because $G_1,\ 0/x,\ 1/y \models \neg E(x,y)$

Thus, $G_1 \models \alpha_1$.

G_2



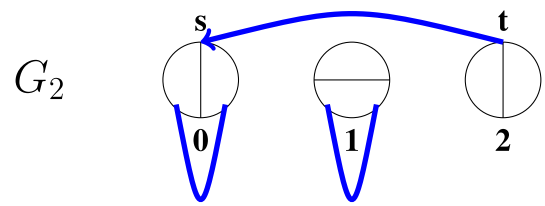
$$G_2 \quad \models \quad \forall xy\,(\neg E(x,y) \;\vee\; E(y,x))$$

$$\text{iff} \quad \bigwedge_{a=0}^{2} G_1, a/x \quad \models \quad \forall y\,(\neg E(x,y) \;\vee\; E(y,x))$$

$$\text{iff} \quad \bigwedge_{a=0}^{2} \bigwedge_{b=0}^{2} G_1, a/x, b/y \quad \models \quad \neg E(x,y) \;\vee\; E(y,x)$$

9 facts to check; one of them is false, which one?

i>Clicker Question 22.1



$$G_2 \models \forall xy\,(\neg E(x,y) \vee E(y,x))$$

$$\text{iff} \quad \bigwedge_{a=0}^{2} G_1, a/x \models \forall y\,(\neg E(x,y) \vee E(y,x))$$

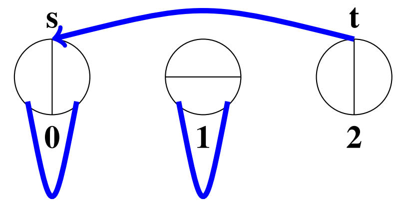
$$\text{iff} \quad \bigwedge_{a=0}^{2}\bigwedge_{b=0}^{2} G_1, a/x, b/y \models \neg E(x,y) \vee E(y,x)$$

Which value of a gives the counterexample?

A : $a = 0$ **B** : $a = 1$ **C** : $a = 2$

i>Clicker Question 22.2

G_2



$$G_2 \quad \models \quad \forall xy\,(\neg E(x,y) \ \vee \ E(y,x))$$

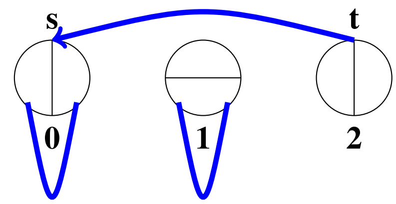
$$\text{iff} \quad \bigwedge_{a=0}^{2} G_1, a/x \quad \models \quad \forall y\,(\neg E(x,y) \ \vee \ E(y,x))$$

$$\text{iff} \quad \bigwedge_{a=0}^{2} \bigwedge_{b=0}^{2} G_1, a/x, b/y \quad \models \quad \neg E(x,y) \ \vee \ E(y,x)$$

Which value of b gives the counterexample?

A : $b \ = \ 0$ **B** : $b \ = \ 1$ **C** : $b \ = \ 2$

G_2



$$G_2 \quad \models \quad \forall xy\,(\neg E(x,y) \ \vee \ E(y,x))$$

$$\text{iff} \quad \bigwedge_{a=0}^{2} G_1, a/x \quad \models \quad \forall y\,(\neg E(x,y) \ \vee \ E(y,x))$$

$$\text{iff} \quad \bigwedge_{a=0}^{2} \bigwedge_{b=0}^{2} G_1, a/x, b/y \quad \models \quad \neg E(x,y) \ \vee \ E(y,x)$$

9 facts to check; one of them is false, which one?

$$G_2,\, 2/x,\, 0/y \quad \not\models \quad \neg E(x,y) \ \vee \ E(y,x)$$

Thus, $G_2 \not\models \alpha_1$.

Truth Game: a two player game that is an equivalent but more fun way to tell whether $G \models \varphi$.

Dumbledore wants to show that $G \models \varphi$.

Gandalf wants to show that $G \not\models \varphi$.

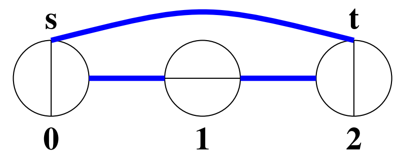
base case: if φ is a literal, then Dumbledore wins iff $G \models \varphi$.

inductive cases:

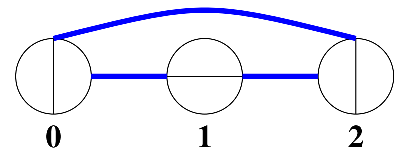
$G \models \varphi \wedge \psi$	**Gandalf**	chooses	$\alpha \in \{\varphi, \psi\}$	continue on:	$G \models \alpha$
$G \models \varphi \vee \psi$	**Dumbledore**	chooses	$\alpha \in \{\varphi, \psi\}$	continue on:	$G \models \alpha$
$G \models \forall x\, \varphi$	**Gandalf**	chooses	$a \in U^G$	continue on:	$G\, a/x \models \varphi$
$G \models \exists x\, \varphi$	**Dumbledore**	chooses	$a \in U^G$	continue on:	$G\, a/x \models \varphi$

$$G_1 \models \forall xy\,(\neg E(x,y) \lor E(y,x))$$

G_1



i>Clicker Question 22.3 $G_1 \models \forall xy\,(\neg E(x,y) \vee E(y,x))$ G_1



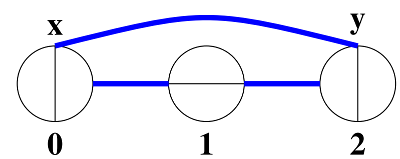
Who makes the first move?

B: Gandalf

It doesn't matter what Gandalf does because all the edges are undirected.

Say, for example, that at this move Gandalf chooses $x = 0$ and at his next move, Gandlaf chooses $y = 2$.

i>Clicker Question 22.4 $G_1\, 0/x,\, 2/y \;\models\; \neg E(x,y) \,\vee\, E(y,x)$ G_1



Who makes the next move?

A: **Dumbledore**

Dumbledore makes this move, choosing the second disjunct, $E(y,x)$

Dumbledore wins the game because $G_1\, 0/x,\, 2/y \models E(y,x)$

Theorem 22.2 (Equivalence of Tarski's Definition of Truth and the Truth Game) *Let Σ be any vocabulary, $\varphi \in \mathrm{FO}[\Sigma]$ and $\mathcal{W} \in \mathrm{WORLD}[\Sigma]$. Then*

$$\text{Dumbledore } \textit{wins the truth game on } \mathcal{W}, \varphi \quad \text{iff} \quad \mathcal{W} \models \varphi$$

$$\text{Gandalf } \textit{wins the truth game on } \mathcal{W}, \varphi \quad \text{iff} \quad \mathcal{W} \models \neg\varphi$$

Proof: By induction on the structure of φ.

base case: φ is a literal. By definition of the truth game

$$\text{Dumbledore wins the game} \quad \Leftrightarrow \quad \mathcal{W} \models \varphi$$

inductive hypothesis: Tarski and Truth Game defs coincide for all smaller formulas, α and β.

- $\varphi = \alpha \wedge \beta$

  By Tarski: $G \models \alpha \wedge \beta$ iff $G \models \alpha$ and $G \models \beta$.

  case 1: $G \models \alpha \wedge \beta$. Thus $G \models \alpha$ and $G \models \beta$.

  By **indHyp** Dumbledore wins on G, α and on G, β. Therefore no matter what Gandalf plays, Dumbledore wins!

  case 2 $G \not\models \alpha \wedge \beta$. Thus $G \not\models \alpha$ or $G \not\models \beta$.

  Gandalf choose one thats false, e.g., $G \not\models \alpha$.

  By **indHyp** Gandalf wins the game on G, α. Therefore Gandalf wins the whole game.

- $\varphi = \exists x\, \alpha$

  By Tarski: $G \models \exists x\, \alpha$ iff for some $a \in U^G$, $G,\ a/x\ \models \alpha$.

  case 1: $G \models \exists x\, \alpha$. Thus $G\, a/x\ \models \alpha$ for some $a \in U^G$. Dumbledore chooses a and they continue to play on $G\, a/x, \alpha$. By **indHyp**, Dumbledore wins the remaining game.

  case 2 $G \not\models \exists x\, \alpha$. Thus no matter which $a \in U^G$, Dumbledore chooses, $G,\ a/x\ \not\models \alpha$. thus, by **indHyp**, Gandalf wins.

The other cases: $\varphi = \alpha \vee \beta$; $\varphi = \forall x\, \alpha$, are similar. □