5.1 Syntax of First Order Logic with Equality

- VAR $\stackrel{\text{def}}{=} \{x, y, z, u, v, w, x_0, y_0, \ldots, x_1, y_1, \ldots\}$
- Vocabulary: $\Sigma = (P_1, P_2, \ldots, P_s; f_1, f_2, \ldots, f_t)$
- Relation Symbols: P_i;   ar(P_i) is its arity, i.e., the number of arguments it takes.
- Function Symbols: f_i of arity ar(f_i), typically abbreviated as f, g, h
- Constant Symbols: f_i when ar$(f_i) = 0$, typically abbreviated as a, b, c, d, k

Definition 5.1 term(Σ) Terms are the strings that represent objects.

Base Case: If $v \in$ VAR then $v \in$ term(Σ).

Inductive Case: If $t_1, t_2, \ldots, t_r \in$ term(Σ), $f \in \Sigma$ and ar$(f) = r$ then $f(t_1, \ldots, t_r) \in$ term(Σ) □

A term $t \in$ term(Σ) is a syntactic object that any structure $\mathcal{A} \in$ STRUC$[\Sigma]$ will have to interpret as an element $t^{\mathcal{A}} \in |\mathcal{A}|$.

Definition 5.2 $L(\Sigma)$ (First Order formulas of Vocab Σ)

Base Case: atomic formulas

If $t_1, \ldots, t_{\text{ar}(P_i)} \in$ term(Σ) and $P_i \in \Sigma$ then $P_i(t_1, \ldots, t_{\text{ar}(P_i)}) \in L(\Sigma)$.

Inductive Steps:

If $\alpha, \beta \in L(\Sigma)$ and $v \in$ VAR then

1. $\neg\alpha \in L(\Sigma)$
2. $(\alpha \vee \beta) \in L(\Sigma)$
3. $\exists v(\alpha) \in L(\Sigma)$

□

Abbreviation: $\forall x(\alpha) \hookrightarrow \neg\exists x(\neg\alpha)$

5.2 Semantics of First Order Logic with Equality

Definition 5.3 $\mathcal{A}$ is a logical structure of vocabulary Σ ($A \in \text{STRUC}[\Sigma]$) iff

$$\mathcal{A} \quad = \quad (|\mathcal{A}|, P_1^{\mathcal{A}}, \ldots, P_s^{\mathcal{A}}; f_1^{\mathcal{A}}, \ldots, f_t^{\mathcal{A}})$$

$|\mathcal{A}| \neq \emptyset$,

$P_i^{\mathcal{A}} \subseteq |\mathcal{A}|^{\text{ar}(P_i)}$, $\mathcal{A}$ interprets predicate symbol P_i as an $\text{ar}(P_i)$-ary relation over its universe.

$f_i^{\mathcal{A}} : |\mathcal{A}|^{\text{ar}(f_i)} \to |\mathcal{A}|$, $\mathcal{A}$ interprets function symbol f_i as a total function taking $\text{ar}(f_i)$ arguments.

To simplify Tarski's Definition of Truth, we assume that every structure $\mathcal{A}$ gives a default value $v^{\mathcal{A}} \in |\mathcal{A}|$ to every variable $v \in \text{VAR}$. □

5.3 Tarski's Definition of Truth

Definition 5.4 Every structure $\mathcal{A} \in \text{STRUC}[\Sigma]$ inteprets every term $t \in \text{term}(\Sigma)$ as an element, $t^{\mathcal{A}}$ of its universe.

base case: If $v \in \text{VAR}$ then $v^{\mathcal{A}} \in |\mathcal{A}|$ is already defined.

inductive case: If $t_1, \ldots, t_r$ are terms already defined by $\mathcal{A}$, and $f^r \in \Sigma$, then

$$f(t_1, \ldots, t_r)^{\mathcal{A}} \quad \stackrel{\text{def}}{=} \quad f^{\mathcal{A}}(t_1^{\mathcal{A}}, \ldots, t_r^{\mathcal{A}})$$ □

Definition 5.5 [Truth] Let $\varphi \in L(\Sigma)$, $\mathcal{A} \in \text{STRUC}[\Sigma]$. We inductively define whether or not $\mathcal{A} \models \varphi$.

Base Case: (Atomic Formulas)

1. $\mathcal{A} \models P_i(t_1, \ldots, t_{\text{ar}(P_i)})$ iff $(t_1^{\mathcal{A}}, \ldots, t_{\text{ar}(P_i)}^{\mathcal{A}}) \in P_i^{\mathcal{A}}$
2. $\mathcal{A} \models t_1 = t_2$ iff $t_1^{\mathcal{A}} = t_2^{\mathcal{A}}$, i.e., we insist that the binary predicate symbol, "=", is always interpreted as "true equality", i.e, $t_1^{\mathcal{A}}$ and $t_2^{\mathcal{A}}$ are the exact same element of $|\mathcal{A}|$. Put another way, $=^{\mathcal{A}} \stackrel{\text{def}}{=} \{(a, a) \mid a \in |\mathcal{A}|\}$.

Inductive Cases:

1. $\mathcal{A} \models \neg\alpha$ iff $\mathcal{A} \not\models \alpha$
2. $\mathcal{A} \models (\alpha \vee \beta)$ iff $\mathcal{A} \models \alpha$ or $\mathcal{A} \models \beta$
3. $\mathcal{A} \models \exists v(\alpha)$ iff there exists $a \in |\mathcal{A}|$ such that $\mathcal{A}[a/v] \models \alpha$

   where $\mathcal{A}[a/v]$ is defined to be the exact same structure as $\mathcal{A}$ with the single exception that the default value of v in $\mathcal{A}[a/v]$ is a, i.e., $v^{\mathcal{A}[a/v]} = a$.

□

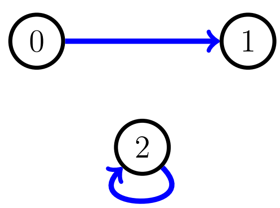
5.4 Examples

Some vocabularies:

- $\Sigma_{\text{graph}} \;=\; (E;),\quad \text{ar}(E) = 2$
- $\Sigma_{\text{st-graph}} \;=\; (E; s, t),\quad \text{ar}(E) = 2, \text{ar}(s) = \text{ar}(t) = 0$
- $\Sigma_{\text{N}} \;=\; (\leq [\text{infix}]; 0, \text{Suc}, +[\text{infix}], *[\text{infix}]),\quad \text{ar}(\leq) = \text{ar}(+) = \text{ar}(*) = 2, \text{ar}(\text{Suc}) = 1$
- $\Sigma_{\text{set}} \;=\; (\in [\text{infix}]; \emptyset),\quad \text{ar}(\in) = 2, \text{ar}(\emptyset) = 0$
- $\Sigma_{\text{group}} \;=\; (; \circ[\text{infix}], e),\quad \text{ar}(\circ) = 2, \text{ar}(e) = 0$

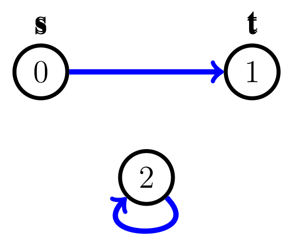
Some structures:

G_0



$E^{G_0} = \{(0,1),(2,2)\}$

G_0'



$s^{G_0'} = 0,\; t^{G_0'} = 1$

$G_0 \in \text{STRUC}[\Sigma_{\text{graph}}];\; G_0' \in \text{STRUC}[\Sigma_{\text{st-graph}}];\; |G_0| = |G_0'| = \{0,1,2\}$

$E(s,t) \;\in\; L(\Sigma_{\text{st-graph}});\quad G_0' \models E(s,t)$

Let $\mathbf{N} = (\{0,1,\ldots\}, \leq^{\mathbf{N}}, \text{Suc}^{\mathbf{N}}, +^{\mathbf{N}}, *^{\mathbf{N}})$. $\mathbf{N}$ is the *standard model of the natural numbers*. $\mathbf{N} \in \text{STRUC}[\Sigma_{\text{N}}]$.

$$\begin{aligned}
\leq^{\mathbf{N}} &= \{(0,0),(0,1),\ldots,(1,1),(1,2),\ldots,(8,9),(8,10),\ldots\} \\
\text{Suc}^{\mathbf{N}} &= \{(0,1),(1,2),(2,3),\ldots\} \\
+^{\mathbf{N}} &= \{((0,0),0),((0,1),1),\ldots((2,2),4)\ldots,((8,9),17)\ldots\} \\
*^{\mathbf{N}} &= \{((0,0),0),((0,1),0),\ldots((2,2),4)\ldots,((8,9),72)\ldots\}
\end{aligned}$$

Let $\alpha \equiv \forall xy(x = y \leftrightarrow \forall z(z \in x \leftrightarrow z \in y))$. α is the "axiom of extensionality", the first axiom of ZFC (Zermelo-Fraenkel plus Choice). It says, "Two sets are equal iff they have exactly the same elements."

A group, G, is a non-empty set with a binary operation that is associative, has an identity and inverses.

The group theory axioms consist of $\gamma_1 \wedge \gamma_2 \wedge \gamma_3$:

$$\begin{array}{ccl} \text{Associative:} & \gamma_1 & \equiv \ \forall xyz\ (x \circ y) \circ z \ = \ x \circ (y \circ z) \\ \text{Identity:} & \gamma_2 & \equiv \ \forall x\ (x \circ e) = x) \\ \text{Inverse:} & \gamma_3 & \equiv \ \forall x \exists y\ (x \circ y) = e) \end{array}$$

A **group** is neither more nor less than a model of the group theory axioms.

A **graph** is neither more nor less than a structure of vocabulary Σ_{graph}.

Let $\psi \in L(\Sigma_{\text{graph}})$ say "loop-free and undirected":

$$\psi \ \equiv \ \forall xy(\neg E(x,x) \ \wedge \ (E(x,y) \rightarrow E(y,x)) \ .$$

A **loop-free, undirected graph** is neither more not less than a model of ψ.