Converting FO Formula to Equivalent Formula in Rectified Prenex Normal (RPF) Form

1. Remove all "$\rightarrow$"s using the fact that $\alpha \rightarrow \beta \;\equiv\; \neg\alpha \vee \beta$.
2. Push all "$\neg$"s all the way inside using de Morgan and quantifier rules:
   - $\neg(\alpha \wedge \beta) \;\equiv\; \neg\alpha \vee \neg\beta$
   - $\neg(\alpha \vee \beta) \;\equiv\; \neg\alpha \wedge \neg\beta$
   - $\neg\forall x(\varphi) \;\equiv\; \exists x(\neg\varphi)$
   - $\neg\exists x(\varphi) \;\equiv\; \forall x(\neg\varphi)$
3. **Rectify** by renaming bound variables so each occurs only once and no bound variable also occurs free.
4. Pull quantifiers out using the following rules, assuming that x does not occur in α:
   - $\alpha \wedge \forall x(\beta) \;\equiv\; \forall x(\alpha \wedge \beta)$
   - $\alpha \vee \forall x(\beta) \;\equiv\; \forall x(\alpha \vee \beta)$
   - $\alpha \wedge \exists x(\beta) \;\equiv\; \exists x(\alpha \wedge \beta)$
   - $\alpha \vee \exists x(\beta) \;\equiv\; \exists x(\alpha \vee \beta)$

Converting an RPF Formula φ to its Skolemization φ_S

- Remove each existential quantifier $\exists x$ and replace each occurrence of x by the term $f(u_1, \ldots, u_r)$ where $u_1, \ldots, u_r$ are the universally quantified variables whose quantifiers occur **before** $\exists x$.

Example:

$$\begin{aligned} \varphi &= \exists x_1 \forall y \exists x_2 \forall z \exists x_3 (R(x_1) \wedge (E(x_1, y) \rightarrow S(y, x_2)) \wedge (E(x_2, z) \rightarrow T(z, x_3))) \\ \varphi_S &= \forall y \forall z (R(a_1) \wedge (E(a_1, y) \rightarrow S(y, f_2(y))) \wedge (E(f_2(y), z) \rightarrow T(z, f_3(y, z)))) \end{aligned}$$

Thm: If φ_S is the Skolemization of φ, then $\varphi \in$ FO-SAT $\Leftrightarrow \varphi_S \in$ FO-SAT.

Note: In using step 4 above in the Prenex Algorithm, you have some choice which quantifier to pull out first, e.g., for $\varphi = (\exists x\, R(x)) \wedge (\forall y\, S(y))$, you may pull out either quantifier first, thus,

$$\varphi \;\equiv\; \exists x\, \forall y\, (R(x)) \wedge S(y)) \;\equiv\; \forall y\, \exists x\, (R(x)) \wedge S(y))\ .$$

When you have this choice, it is better to pull the $\exists$'s out in front of the $\forall$'s. This way, your Skolem functions won't needlessly take irrelevant universally quantified variables as arguments. In the above example, $\varphi_S \equiv \forall y(R(c) \wedge S(y))$, is the preferred Skolemization of φ. $\varphi'_S \equiv \forall y(R(f(y)) \wedge S(y))$ is also correct, but less convenient.