# 1 PropCalc Definitions and Abbreviations

## 1.1 Syntax of PropCalc

$$\mathbf{P}_{\mathbf{var}} \stackrel{\text{def}}{=} \{p, q, r, s, p_0, q_0, r_0, s_0, p_1, q_1, r_1, s_1, \ldots\}$$ **PropCalc Variables**

**Definition 1.1** [Inductive Definition of Propositional Formulas ($\mathbf{P}_{\mathbf{fmla}}$) (Syntax of PropCalc)

**base 0.** $\top, \bot \in \mathbf{P}_{\mathbf{fmla}}$

**base 1.** If $a \in \mathbf{P}_{\mathbf{var}}$ then $a \in \mathbf{P}_{\mathbf{fmla}}$

**inductive 2.** If $\alpha \in \mathbf{P}_{\mathbf{fmla}}$ then $\neg\alpha \in \mathbf{P}_{\mathbf{fmla}}$

**inductive 3.** If $\alpha, \beta \in \mathbf{P}_{\mathbf{fmla}}$ then $(\alpha \vee \beta) \in \mathbf{P}_{\mathbf{fmla}}$

□

## 1.2 Precedence

1. $\neg$ binds most tightly, then
2. $\wedge, \vee$, then
3. $\rightarrow, \leftrightarrow$
4. $\rightarrow$ associates as follows: $\alpha \rightarrow \beta \rightarrow \gamma \quad \equiv \quad \alpha \rightarrow (\beta \rightarrow \gamma)$

Example: $\neg\alpha \wedge \beta \rightarrow \neg\gamma \vee \delta \rightarrow \epsilon \quad \equiv \quad (\neg\alpha \wedge \beta) \rightarrow ((\neg\gamma \vee \delta) \rightarrow \epsilon)$

## 1.3 Semantics of PropCalc

A **truth assignment** or **world** is a function $\mathcal{A} : D \rightarrow \{0, 1\}$ where $\text{dom}(\mathcal{A}) = D \subseteq \mathbf{P}_{\mathbf{var}}$.

Extend $\mathcal{A}$ to $\mathcal{A}' : \mathbf{P}_{\mathbf{fmla}}(D) \rightarrow \{0, 1\}$ inductively as follows:

1. $\mathcal{A}(\top) \stackrel{\text{def}}{=} 1$; $\mathcal{A}(\bot) \stackrel{\text{def}}{=} 0$
2. for $v \in D$, $\mathcal{A}'(v) \stackrel{\text{def}}{=} \mathcal{A}(v)$
3. $\mathcal{A}'(\neg\alpha) \stackrel{\text{def}}{=} 1 - \mathcal{A}'(\alpha)$
4. $\mathcal{A}'(\alpha \vee \beta) \stackrel{\text{def}}{=} \max(\mathcal{A}'(\alpha), \mathcal{A}'(\beta))$

[For convenience, we will assume that $\mathcal{A} = \mathcal{A}'$, i.e., don't bother writing the " $'$ ".]

We say that $\mathcal{A}$ is **suitable** or **appropriate** for $\alpha$ if $\text{var}(\alpha) \subseteq \text{dom}(\mathcal{A})$.

Example: if $\text{dom}(\mathcal{A}) = \{p, q\}$, then $\mathcal{A}$ is suitable for $p \vee \neg p \to \top$, but not for $\bot \to q \vee r$.

Note that a **Truth Table** can provide an equivalent definition for the semantics of PropCalc.

| truth assignment | $p$ | $q$ | $\neg p$ | $p \vee q$ | $p \wedge q$ | $p \to q$ |
|---|---|---|---|---|---|---|
| $\mathcal{A}_0$ | 0 | 0 | 1 | 0 | 0 | 1 |
| $\mathcal{A}_1$ | 0 | 1 | 1 | 1 | 0 | 1 |
| $\mathcal{A}_2$ | 1 | 0 | 0 | 1 | 0 | 0 |
| $\mathcal{A}_3$ | 1 | 1 | 0 | 1 | 1 | 1 |

**Note:** If $\alpha \in \mathbf{P_{fmla}}$ has $n$ variables, then there are $2^n$ possible truth assignments of interest. These are all the different worlds that might exist from $\alpha$'s point of view.

## 1.4 Equivalence

**Definition 1.2** [**Semantic Implication**] For a set of formula, $\Gamma \subseteq \mathbf{P_{fmla}}$, and world, $\mathcal{A}$, we write $\mathcal{A} \models \Gamma$ to mean that $\mathcal{A}$ satisifies every formula in $\Gamma$. For $\alpha \in \mathbf{P_{fmla}}$, we write $\Gamma \models \alpha$ ( $\Gamma$ **semantically implies** $\alpha$) to mean that for all appropritate $\mathcal{A}$, If $\mathcal{A} \models \Gamma$, then $\mathcal{A} \models \alpha$. This is called semantic implication because it is a form of implication having to do with semantics, i.e., what worlds $\Gamma$ and $\alpha$ hold in. Once we prove the completeness theorem, i.e., $(\Gamma \models \alpha) \Leftrightarrow (\Gamma \vdash \alpha)$, we will use $\models$ just for satisfaction, i.e., $\mathcal{A} \models \alpha$ ($\alpha$ is true in $\mathcal{A}$) and $\Gamma \vdash \alpha$ ($\Gamma$ proves $\alpha$), i.e., there is a resolution proof of $\alpha$, using assumptions from $\Gamma$. $\square$

**Definition 1.3** For $\alpha, \beta \in \mathbf{P_{fmla}}$, we say that $\alpha$ and $\beta$ are **equivalent** iff $\alpha \models \beta$ and $\beta \models \alpha$, i.e., for all $\mathcal{A}$ appropritate for $\alpha$ and $\beta$, $\mathcal{A}(\alpha) = \mathcal{A}(\beta)$. $\square$

**Some Important Equivalences (worth memorizing):**

| | | | |
|---|---|---|---|
| **contrapositive** | $p \to q$ | $\equiv$ | $\neg q \to \neg p$ |
| **de Morgan** | $\neg(p \vee q)$ | $\equiv$ | $\neg p \wedge \neg q$ |
| **de Morgan** | $\neg(p \wedge q)$ | $\equiv$ | $\neg p \vee \neg q$ |
| **excluded middle** | $p \vee \neg p$ | $\equiv$ | $\top$ |
| **double negation** | $p$ | $\equiv$ | $\neg\neg p$ |
| **commutative laws** | $p \vee q$ | $\equiv$ | $q \vee p$ |
| **commutative laws** | $p \wedge q$ | $\equiv$ | $q \wedge p$ |
| **associative laws** | $(p \vee q) \vee r$ | $\equiv$ | $p \vee (q \vee r)$ |
| **associative laws** | $(p \wedge q) \wedge r$ | $\equiv$ | $p \wedge (q \wedge r)$ |
| **distributive laws** | $p \vee (q \wedge r)$ | $\equiv$ | $(p \vee q) \wedge (p \vee r)$ |
| **distributive laws** | $p \wedge (q \vee r)$ | $\equiv$ | $(p \wedge q) \vee (p \wedge r)$ |

## 1.5 Abbreviations

**"$\hookrightarrow$" is an abbreviation for "is an abbreviation for"**

$$\alpha \wedge \beta \hookrightarrow \neg(\neg\alpha \vee \neg\beta)$$

$$\alpha \to \beta \hookrightarrow \neg\alpha \vee \beta$$

$$a \leftrightarrow \beta \hookrightarrow (a \to \beta) \wedge (\beta \to a)$$

(See **abbreviations.pdf** on the syllabus page:

`https://people.cs.umass.edu/~immerman/cs513/syllabus.html`

where I post important abbreviations.)