Name: ______________________________

CS601:	**Final**	a previous year

This is a **closed book and notes** exam, but our crib sheet and complexity class diagram are on the last two pages. Please read all four parts of the first question and both parts of the second question.

Please write your answers right on this test, using the back if necessary

Please try to be clear and only write things that are true. Partial credit will be given for partial solutions.

Good luck!

1. [68 pts.] Each of the following four problems is complete for some complexity class among NL, P, NP, co-NP, PSPACE, EXPTIME, NEXPTIME, r.e., co-r.e. For each problem, determine which complexity class it is complete for and prove that it is complete for this class. (This involves showing that it is in the class and hard for the class.)

   (a) $S_a = \left\{ m\#w\#^r \;\middle|\; M_m \text{ accepts } w \text{ using at most } r \text{ work-tape cells} \right\}$

(b) $S_b = \{D \mid D \text{ is a DFA and } \mathcal{L}(D) \text{ is infinite}\}$

(c) $S_c = \left\{ m \;\middle|\; \text{there is no prime number, } p, \text{ s.t. } M_m(p) = 1 \right\}$

[In this problem, P is Cantor's pairing function.]

(d) $S_d \;=\; \left\{ P(m,x) \in \mathbf{N} \;\middle|\; (\exists y \leq x)(M_m(P(y,x))\text{ halts within } |x|^2 \text{ steps }) \right\}$

2. [32 pts.] For each of the following two first-order formulas, either it is valid, or it's unsatisfiable, or neither of these is true. If it is valid give a first-order proof, if it is unsatisfiable give a first-order proof of its negation. Your first-order proof may use any of our meta rules. If the formula and its negation are both satisfiable, describe two logical structures: one that satisfies the formula and one that satisfies its negation.

   [Hint and example: the formula $(P(x) \vee \neg P(x))$ is valid and it has a one line proof because it is an instance of Axiom 0. The formula $\neg((P(x) \vee \neg P(x))$ is unsatisfiable and it's negation, $\neg\neg((P(x) \vee \neg P(x))$ is also an instance of Axiom 0. The formula $P(x)$ is neither. Let $\mathcal{A}_0$ be the structure with universe $\{a, b\}$ and with $P^{\mathcal{A}_0} = \{a\}$. Then $(\mathcal{A}_0, a/x) \models P(x)$ and $(\mathcal{A}_0, b/x) \models \neg P(x)$.]

   (a) $\varphi_{2a} \equiv \forall x(P(x) \rightarrow \exists y(Q(x, y))) \rightarrow \forall x \exists y(P(x) \rightarrow Q(x, y))$

(b) $\varphi_{2b} \quad \equiv \quad \forall x \exists y (P(x) \rightarrow Q(x,y)) \ \rightarrow \ \exists y \forall x (P(x) \rightarrow Q(x,y))$

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CS601	**Final Crib Sheet**	a previous year

Regular and CFL Pumping Lemmas::

A regular $\Rightarrow\ \exists n \forall w \exists x,y,z((w \in A \wedge |w| \geq n) \rightarrow (w = xyz\ \wedge\ |xy| \leq n\ \wedge\ |y| > 0\ \wedge\ (\forall i \geq 0)xy^iz \in A))$

A CFL$\Rightarrow \exists n \forall z \exists u,v,w,x,y((z \in A \wedge |z| \geq n) \rightarrow (z = uvwxy \wedge |vx| > 0 \wedge |vwx| \leq n \wedge (\forall i \geq 0)uv^iwx^iy \in A))$

COMP $\in$ **PrimRecFcns** $=$ Bloop Fcns: COMP$(n,x,c,y) \Leftrightarrow$ (c is the computation of $M_n(x) = y$)

Recursive $=$ **r.e.** $\cap$ **co-r.e.** $\quad W_i = \{n \mid M_i(n) = 1\}$; $K = \{n \mid M_n(n) = 1\}$

Tarski's Inductive Definition of Truth:

$(\mathcal{A}, i) \models t_1 = t_2 \Leftrightarrow i^\star(t_1) = i^\star(t_2)$

$(\mathcal{A}, i) \models \neg\varphi \Leftrightarrow (\mathcal{A}, i) \not\models \varphi$

$(\mathcal{A}, i) \models \forall x(\varphi) \Leftrightarrow$ (for all $a \in |\mathcal{A}|)(\mathcal{A}, i, a/x) \models \varphi$

$(\mathcal{A}, i) \models R_j(t_1, \ldots, t_{a_j}) \Leftrightarrow \langle i^\star(t_1), \ldots, i^\star(t_{a_j})\rangle \in R_j^{\mathcal{A}}$

$(\mathcal{A}, i) \models \varphi \vee \psi \Leftrightarrow (\mathcal{A}, i) \models \varphi$ or $(\mathcal{A}, i) \models \psi$

FO Axioms: all generalizations of the following:	
0	Tautologies on **at most three** boolean variables
1a	$t = t$
1b	$(t_1 = t_1' \wedge \cdots \wedge t_k = t_k') \rightarrow f(t_1, \ldots, t_k) = f(t_1', \ldots, t_k')$
1c	$(t_1 = t_1' \wedge \cdots \wedge t_k = t_k') \rightarrow (R(t_1, \ldots, t_k) \rightarrow R(t_1', \ldots, t_k'))$
2	$\forall x(\varphi) \rightarrow \varphi[x \leftarrow t]$
3	$\varphi \rightarrow \forall x(\varphi)$, x not free in φ
4	$\forall x(\varphi \rightarrow \psi) \rightarrow (\forall x(\varphi) \rightarrow \forall x(\psi))$

Modus Ponens (M.P.):

$$\frac{\Gamma \vdash \varphi \rightarrow \psi,\ \Gamma \vdash \varphi}{\Gamma \vdash \psi}$$

Meta Rules:

$\forall$ **Intro:** If $\Gamma \vdash \varphi$ and x does not occur freely in Γ, Then $\Gamma \vdash \forall x(\varphi)$.

$\forall$ **Elim:** If $\Gamma \vdash \forall x(\varphi)$ Then $\Gamma \vdash \varphi[x \leftarrow t]$. $\rightarrow$ **Intro:** If $\Gamma \cup \{\varphi\} \vdash \psi$, Then $\Gamma \vdash \varphi \rightarrow \psi$.

$\exists$ **Elim:** If $\Gamma \vdash \exists x(\varphi)$, and $\Gamma, \varphi[x \leftarrow c] \vdash \psi$, where c does not occur in Γ, φ, or ψ, Then $\Gamma \vdash \psi$.

$\exists$ **Intro:** If $\Gamma \vdash \varphi[x \leftarrow t]$ Then $\Gamma \vdash \exists x(\varphi)$ $\vee$ **Intro:** $\{\alpha\} \vdash \alpha \vee \beta$; $\{\alpha\} \vdash \beta \vee \alpha$

$\vee$ **Elim:** If $\Gamma \vdash \alpha \vee \beta$; $\Gamma \cup \{\alpha\} \vdash \gamma$; and $\Gamma \cup \{\beta\} \vdash \gamma$, Then $\Gamma \vdash \gamma$

$\wedge$ **Intro:** $\{\alpha, \beta\} \vdash \alpha \wedge \beta$; $\wedge$ **Elim:** $\{\alpha \wedge \beta\} \vdash \alpha$; $\{\alpha \wedge \beta\} \vdash \beta$

Proof by Contradiction: If $\Gamma \cup \{\neg\varphi\} \vdash \bot$, Then $\Gamma \vdash \varphi$.

Complete for: **NL:** REACH, 2-SAT, DFA-Empty, NFA-Empty

P: CVP, MCVP, Horn-SAT, AREACH, CFL-Empty

NP: TSP, SAT, 3-SAT, 3-COLOR, CLIQUE, Subset Sum, Knapsack, Unary Tiling

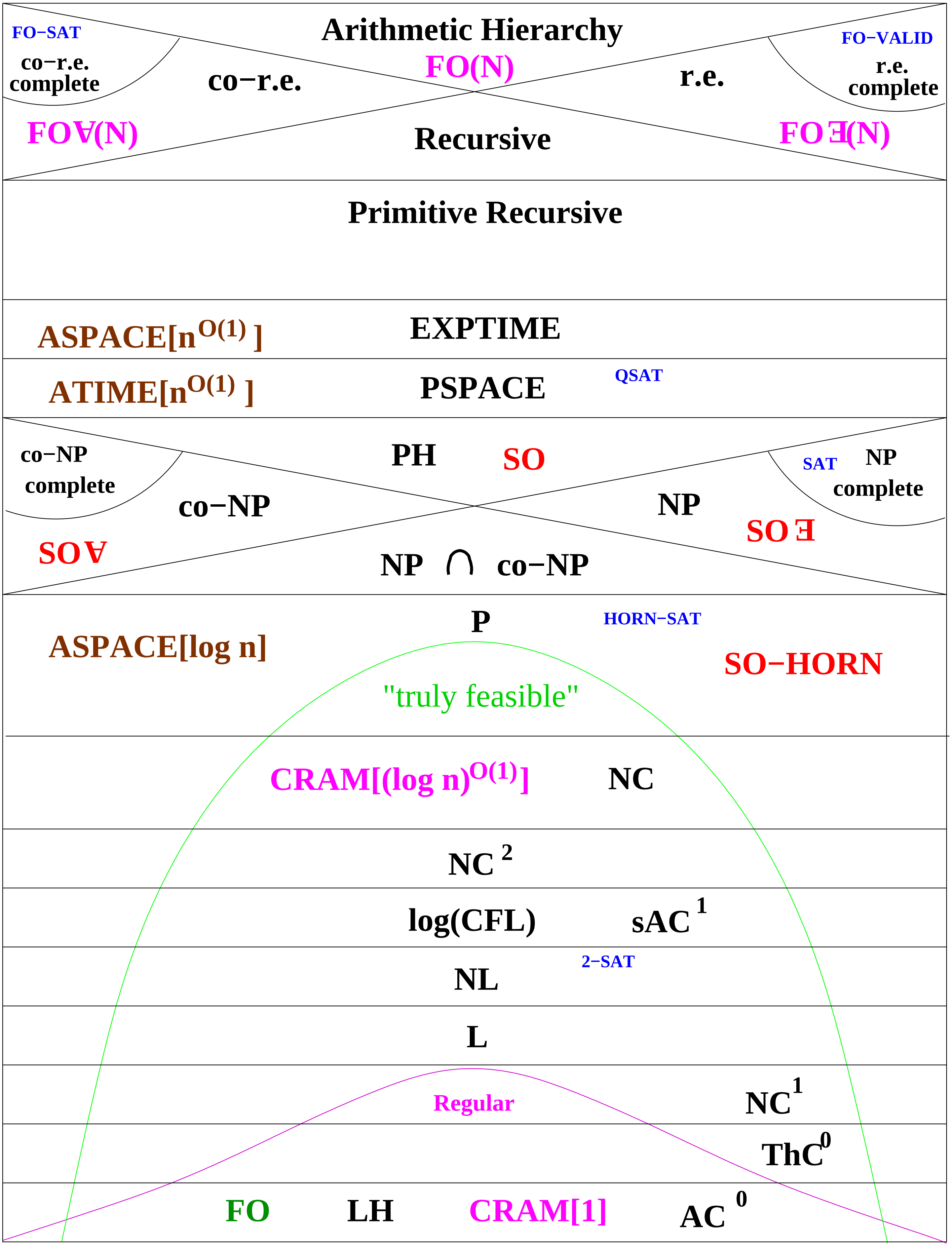
PSPACE: QSAT, GEOGRAPHY, Succinct-REACH, NFA-$\Sigma^\star$

NEXPTIME: Tiling

r.e.: K, HALT, FO-VALID; **co-r.e.:** $\overline{K}$, $\Sigma^\star$CFL, TM-EMPTY, FO-SAT

Savitch & CKS Ths: For $s(n) \geq \log n$, NSPACE$[s(n)] \subseteq$ ATIME$[(s(n))^2] \subseteq$ DSPACE$[(s(n))^2]$;

ASPACE$[s(n)] =$ DTIME$[2^{O(s(n))}]$


Arithmetic Hierarchy
FO−SAT
co−r.e. complete
co−r.e.
FO(N)
r.e.
FO−VALID
r.e. complete
FO∀(N)
Recursive
FO∃(N)
Primitive Recursive
ASPACE[n^O(1)]
EXPTIME
ATIME[n^O(1)]
PSPACE
QSAT
co−NP complete
PH
SO
SAT
NP complete
co−NP
NP
SO∃
SO∀
NP ∩ co−NP
P
HORN−SAT
ASPACE[log n]
SO−HORN
"truly feasible"
CRAM[(log n)^O(1)]
NC
NC 2
log(CFL)
sAC 1
NL
2−SAT
L
Regular
NC 1
ThC 0
FO
LH
CRAM[1]
AC 0