CS601:	**Homework 5**	Due in main office before 4:30 pm, March, 24, 2009

Problems:

1. [30 pts.] Let $\mathcal{PF} = \{M_i(\cdot) \mid i \in \mathbf{N}\}$ be the set of partial, recursive functions. For any subset, $\mathcal{S} \subseteq \mathcal{PF}$, let $\mathrm{PROG}(\mathcal{S}) = \{i \mid M_i(\cdot) \in \mathcal{S}\}$ be the set of programs of TM's that compute partial functions in $\mathcal{S}$. Suppose that $\emptyset \subsetneq \mathcal{S} \subsetneq \mathcal{PF}$. Prove that $\mathrm{PROG}(\mathcal{S})$ is not recursive. In particular, let $f_\emptyset$ be the empty partial function[1], i.e., the function that is defined nowhere. Prove the following:

   (a) If $f_\emptyset \in \mathcal{S}$ then $\overline{K} \leq \mathrm{PROG}(\mathcal{S})$.

   (b) If $f_\emptyset \notin \mathcal{S}$ then $K \leq \mathrm{PROG}(\mathcal{S})$.

2. [30 pts.] Let Γ_0 be a set of boolean expressions over the boolean variables, $X = \{x_1, x_2, x_3, \ldots\}$, as defined in Lecture 10, slide 5. Assume that every finite subset of Γ_0 is satisifiable. In this problem you will prove the Compactness Theorem for Boolean Expressions, i.e., if every finite subset of Γ_0 is satisfiable, then Γ_0 is satisfiable.

   (a) Let $x_i \in X$ be a boolean variable, and let S be a set of boolean expressions such that every finite subset of S is satisfiable. Show that the same is true of at least one of $S \cup \{x_i\}$, and $S \cup \{\neg x_i\}$.

   (b) Inductively define,

   $$\Gamma_{i+1} \;=\; \begin{cases} \Gamma_i \cup \{x_i\} & \text{if every finite subset of this set is satisfiable} \\ \Gamma_i \cup \{\neg x_i\} & \text{otherwise} \end{cases}$$

   Show by induction that for all i, every finite subset of Γ_i is satisfiable.

   (c) Define $\Delta = \bigcup_{i=0}^{\infty} \Gamma_i$.

   Show that every finite subset of Δ is satisfiable.

   (d) Prove that Δ is satisfiable. [Hint: Δ knows how it feels about every boolean variable.]

[1] Actually, a function is a set of ordered pairs, so the empty function is the set of no pairs, i.e. it is actually equal to the emptyset, i.e., $f_\emptyset = \emptyset$.

3. [40 pts.] This problem concerns the Arithmetic Hierarchy, which is at the top of the World-of-Computability-and-Complexity diagram. We say that a set of natural numbers, S, is an element of Σ_k iff there is a primitive-recursive predicate φ, such that,

$$S \quad = \quad \{n \mid \exists x_1 \forall x_2 \cdots Q_k x_k(\varphi(n, x_1, \ldots, x_k))\},$$

here Q_k is $\forall$ if k is even and $\exists$ if k is odd. Similarly, S is an element of Π_k iff,

$$S \quad = \quad \{n \mid \forall x_1 \exists x_2 \cdots Q'_k x_k(\psi(n, x_1, \ldots, x_k))\},$$

for some primitive-recursive predicate ψ. Here Q'_k is $\forall$ if k is odd and $\exists$ if k is even. Define the Arithmetic Hierarchy to be $\bigcup_{k=1}^{\infty} \Sigma_k$.

For example, $K \in \Sigma_1$, EMPTY $\in \Pi_1$, and TOTAL $\in \Pi_2$ as follows:

$$\begin{array}{rcccl}
K & = & \{n \mid M_n(n) = 1\} & = & \{n \mid \exists x(\text{COMP}(n, n, x, 1))\} \\
\text{EMPTY} & = & \{n \mid W_n = \emptyset\} & = & \{n \mid \forall x(\neg\text{COMP}(n, L(x), R(x), 1))\} \\
\text{TOTAL} & = & \{n \mid M_n(\cdot) \text{ is a total function}\} & = & \{n \mid \forall x \exists y(\text{COMP}(n, x, L(y), R(y)))\}
\end{array}$$

(a) Prove that r.e. = Σ_1, i.e., a set is r.e. iff it is in Σ_1.

(b) Prove that co-r.e. = Π_1.

(c) Classify the following sets by writing a formula that places them as low as you can in the arithmetic hierarchy. For example, You do not have to prove that they cannot be placed in a lower class, but do your best. Points will be taken off for solutions that are higher in the hierarchy than they need to be.

   i. $A_{0,17} = \{n \mid M_n(0) = 17\}$
   ii. $S_2 = \{n \mid M_n(\mathbf{N}) \subseteq \{17\}\}$
   iii. $S_3 = \{n \mid M_n(\mathbf{N}) = \{17\}\}$
   iv. INITIAL-SEGMENT $= \{n \mid W_n = \{0, 1, \ldots, k\}$ for some $k \in \mathbf{N}\}$