CS601:	**Homework 6**	Due in main office before 4:30 pm, March, 31, 2009

Problems:

1. [25 pts.] For each of the following three formulas, φ_i, produce structures $\mathcal{A}_i, \mathcal{B}_i$ such that $\mathcal{A} \models \varphi_i$ and $\mathcal{B} \models \neg\varphi_i$, and very briefly explain why these properties hold.

   (a) $\varphi_1 \ \equiv\ \forall x(A(x) \rightarrow B(x))$

   (b) $\varphi_2 \ \equiv\ \forall x\, \forall y\, (f(x) = f(y) \rightarrow x = y)$

   (c) $\varphi_3 \ \equiv\ \forall y\, \exists x\, P(x, y) \rightarrow \exists x\, \forall y\, P(x, y)$

2. [25 pts.] Consider the following three first-order sentences:

   $$\rho \ \equiv\ (\forall x) P(x, x) \qquad \text{“}P \text{ is } \textbf{reflexive}\text{”}$$
   $$\sigma \ \equiv\ (\forall xy)(P(x, y) \rightarrow P(y, x)) \qquad \text{“}P \text{ is } \textbf{symmetric}\text{”}$$
   $$\tau \ \equiv\ (\forall xyz)((P(x, y) \wedge P(y, z)) \ \rightarrow\ P(x, z)) \qquad \text{“}P \text{ is } \textbf{transitive}\text{”}$$

   Show that these three sentences are *independent* by producing three structures satisfying the following,

   (a) $\rho \wedge \sigma \wedge \neg\tau$

   (b) $\rho \wedge \neg\sigma \wedge \tau$

   (c) $\neg\rho \wedge \sigma \wedge \tau$

3. [20 pts.] Prove the Lemma on slide 20 of Lecture 12, i.e., let Σ be a vocabulary and $\varphi \in \mathcal{L}(\Sigma)$. Then for all structures, $\mathcal{A}, \mathcal{A}' \in$ STRUC$[\Sigma]$ such that $\mathcal{A}$ and $\mathcal{A}'$ are identical except in how they interpret variables not free in φ,

   $$\mathcal{A} \models \varphi \quad \Leftrightarrow \quad \mathcal{A}' \models \varphi\ .$$

   [Hint: you should prove this by induction on φ, as outlined on slide 20.]

4. [30 pts.] This problem studies the relationships between elementary equivalence and isomorphism, as described in slides 7 and 8 of Lecture 12.

   (a) Show that if $\mathcal{A} \cong \mathcal{B}$ then $\mathcal{A} \equiv \mathcal{B}$. [Hint: to do this, I suggest that you assume that $\mathcal{A} \cong \mathcal{B}$ and then show by induction on φ that $\mathcal{A} \models \varphi \ \Leftrightarrow\ \mathcal{B} \models \varphi$. Using problem 3, you may assume that for all variables, v, $\eta(v^{\mathcal{A}}) = v^{\mathcal{B}}$. Start by showing by induction that the same holds for every term, t.]

   (b) Show that if $\mathcal{A} \equiv \mathcal{B}$ and $|\mathcal{A}|$ is finite then $\mathcal{A} \cong \mathcal{B}$. [Hint: show how to write a complete description of $\mathcal{A}$ in first-order logic. For simplicity, you may assume that Σ is finite.]