CS601:	**Homework 7**	Due in main office before 4:30 pm, April 7, 2009

Problems:

1. [10 pts.] Please transform the following formula to an equivalent formula in prenex normal form. Please show enough intermediate steps so that we can follow your work. If you don't know how to do this, see

   `www.cs.umass.edu/~immerman/cs601/prenexAlgorithm.pdf`.

$$\begin{array}{rcl}\alpha & \equiv & \forall xy((x = y \vee E(x,y) \rightarrow P(x,y)) \ \wedge \ \forall z(P(x,z) \wedge P(z,y) \rightarrow P(x,y))) \\ & \rightarrow & \forall xyz(E(x,y) \wedge E(y,z) \rightarrow P(x,z))\end{array}$$

2. [25 pts.] A first-order formula is **universal** if it is in prenex normal form and all of it's quantifiers are $\forall$'s. Similarly it is **existential** if it is in prenex normal form and all of it's quantifiers are $\exists$'s. Suppose that $\mathcal{A} \leq \mathcal{B}$, i.e. $\mathcal{A}$ is a substructure of $\mathcal{B}$. Prove the following:

   (a) If φ is quantifier-free then $\mathcal{B} \models \varphi$ iff $\mathcal{A} \models \varphi$.

   (b) If φ is universal and $\mathcal{B} \models \varphi$ then $\mathcal{A} \models \varphi$.

   (c) If φ is existential and $\mathcal{A} \models \varphi$ then $\mathcal{B} \models \varphi$.

   [Hint: (b) and (c) imply each other. Just prove one and show why the other follows. In the definition of substructure, for any variable, x, that occurs freely in φ, you should assume that $x^{\mathcal{A}} = x^{\mathcal{B}}$.]

3. [25 pts.] Define the quantifier rank of a first-order formula, qr(φ) inductively as follows:

   - base case: atomic formula, qr$(R(t_1, \ldots, t_k)) = 0$
   - qr$(\neg\varphi) =$ qr(φ)
   - qr$(\varphi \vee \psi) = \max(qr(\varphi),$ qr$(\psi))$
   - qr$(\forall v(\varphi) = 1 +$ qr(φ).

   (a) Let Σ be a finite relational vocabulary, i.e., it has no function symbols of arity greater than 0. Prove by induction that for all k there are only finitely many formulas – up to logical equivalence – of quantifier rank k, and having at most c free variables, for some constant, c.

   (b) Show that if Σ has one function symbol of arity one, then there are infinitely many inequivalent formulas of quantifier rank 0 having one free variable.

4. [40 pts.] Show that each of the following formulas is a theorem of first order logic by giving a detailed proof in our system. You may use the Meta Theorems from Lecture 13. Each line of each proof should be justified as a particular Axiom, MP, Meta Theorem, or a previously proved theorem, as in the examples from Lecture 13, slide 5, and slide 16. In those proofs, I keep track of assumptions explicitly by writing them on the left side of the proves symbol ($\vdash$). It is fine if instead you want to open subproof blocks with given assumptions.

   See the following sheet for a summary of natural deduction rules which may be used – if desired – as meta rules.

   `www.cs.umass.edu/~immerman/cs601/naturalDeductionRules.pdf`

   (a) $\forall x\, \exists y\, (f(x) = y)$

   (b) $\exists x\, \forall y\, P(x, y) \to \forall y\, \exists x\, P(x, y)$

   (c) $(\alpha \vee \forall x(\beta)) \to \forall x(\alpha \vee \beta)$, where x does not occur freely in α.

   (d) $\forall x(\alpha \vee \beta) \to (\alpha \vee \forall x(\beta))$, where x does not occur freely in α.