CS601:	**Homework 8**	Due in main office before 4:30 pm, April 14, 2009

Problems:

1. [25 pts.] A first-order formula is a Σ_2**-formula** if it can be written in prenex form as $\exists x_1 \ldots x_s \forall y_1 \ldots y_t(\alpha)$ where α is quantifier-free. Prove that if we restrict our attention to vocabularies with **no function symbols** then Σ_2-FO-SAT, the set of satisfiable Σ_2 formulas, is a recursive set. [Hint: show that any satisfiable formula of the above form has a model whose universe has at most s elements. Problem 2b from HW 7 is helpful here.]

2. [25 pts.] Let Σ be a first-order alphabet.

   (a) Prove that there is a set of first-order formulas over Σ, Φ_{inf}, such that $\text{MOD}(\Phi_{\text{inf}})$ is exactly the set of all infinite Σ-structures.

   (b) Prove that there is no set of first-order formulas over Σ, Φ, such that $\text{MOD}(\Phi)$ is exactly the set of all finite Σ-structures. [Hint: Use the compactness theorem.]

3. [25 pts.] Prove that NP $\neq$ DSPACE$[n]$. [This is problem 7.4.7 from p. 155 of Papadimitriou. Hint: assume that NP $=$ DSPACE$[n]$ and prove that the DSPACE Hierarchy Theorem fails.]

4. [25 pts.] Let Σ be a fixed, finite vocabulary. Prove that for any first-order sentence, $\varphi \in \mathcal{L}(\Sigma)$, $\text{MOD}_{\text{fin}}[\varphi] \in \text{DSPACE}[\log n]$. Where, $\text{MOD}_{\text{fin}}[\varphi]$ is the set of finite models of φ,

$$\text{MOD}_{\text{fin}}[\varphi] \quad = \quad \{\mathcal{A} \in \text{STRUC}_{\text{fin}}[\Sigma] \mid \mathcal{A} \models \varphi\} \; .$$

   [Hint: prove this by induction on φ. I suggest that you do it first for a rather simple vocabulary, e.g., Σ_g – the vocabulary of graphs – would be good. You can assume that for any graph, G, on n vertices, the TM is given a binary encoding, $\text{bin}(G)$ consisting of $n^2 + 2\lceil \log n \rceil$ bits listing the adjacency matrix for E^G followed by s^G and t^G.]