

# Recall Tarski's Definition of Truth

$$\mathcal{A} = (|\mathcal{A}|, \mu) \in \text{STRUC}[\Sigma]; \quad \varphi \in \mathcal{L}(\Sigma)$$

$\mu$  defined on VAR;

extend to terms

inductively:

$$\mu : x \mapsto x^{\mathcal{A}}$$

$$\mu^* : t \mapsto t^{\mathcal{A}}$$

$$f_j(t_1, \dots, t_{r(f_j)})^{\mathcal{A}} = f_j^{\mathcal{A}}(t_1^{\mathcal{A}}, \dots, t_{r(f_j)}^{\mathcal{A}})$$

$$\mathcal{A} \models t_1 = t_2 \Leftrightarrow t_1^{\mathcal{A}} = t_2^{\mathcal{A}}$$

$$\mathcal{A} \models R_j(t_1, \dots, t_{r(R_j)}) \Leftrightarrow \langle t_1^{\mathcal{A}}, \dots, t_{r(R_j)}^{\mathcal{A}} \rangle \in R_j^{\mathcal{A}}$$

$$\mathcal{A} \models \neg\varphi \Leftrightarrow \mathcal{A} \not\models \varphi$$

$$\mathcal{A} \models \varphi \vee \psi \Leftrightarrow \mathcal{A} \models \varphi \text{ or } \mathcal{A} \models \psi$$

$$\mathcal{A} \models \forall x(\varphi) \Leftrightarrow \text{for all } a \in |\mathcal{A}| \ (\mathcal{A}, a/x) \models \varphi$$

$$\text{where } y^{(\mathcal{A}, a/x)} = \begin{cases} y^{\mathcal{A}} & \text{if } y \neq x \\ a & \text{if } y = x \end{cases}$$

# FO AXIOMS: all generalizations of the following

**AX0:** Tautologies on at most three boolean variables

**AX1a:**  $t = t$  for any term  $t$

**AX1b:**  $(t_1 = t'_1 \wedge \dots \wedge t_k = t'_k) \rightarrow f(t_1, \dots, t_k) = f(t'_1, \dots, t'_k)$   
for terms  $t_1, \dots, t'_k$   $f \in \Phi$ ,  $r(f) = k$

**Ax1c:**  $(t_1 = t'_1 \wedge \dots \wedge t_k = t'_k) \rightarrow (R(t_1, \dots, t_k) \rightarrow R(t'_1, \dots, t'_k))$   
for terms  $t_1, \dots, t'_k$ ,  $R \in \Pi$ ,  $r(R) = k$

**AX2:**  $\forall x(\varphi) \rightarrow \varphi[x \leftarrow t]$   $t$  substitutable for  $x$  in  $\varphi$

**AX3:**  $\varphi \rightarrow \forall x(\varphi)$  where  $x$  does not occur freely in  $\varphi$

**AX4:**  $\forall x(\varphi \rightarrow \psi) \rightarrow (\forall x(\varphi) \rightarrow \forall x(\psi))$

# Theorems and Proofs

**Prop:** Every instance of every axiom is valid.

**Def:** A **proof from assumptions**,  $\Gamma$ , is a sequence of formulas:  $\varphi_0, \varphi_1, \dots, \varphi_n$ , s.t.

Each  $\varphi_i$  is either

- an **axiom**, or
- an **assumption**  $\varphi_i \in \Gamma$ , or
- **follows from previous**  $\varphi_j, \varphi_k$ ,  $j, k < i$  **via M.P.**

If  $\varphi$  is in a proof from  $\Gamma$ , write  $\Gamma \vdash \varphi$ .

If  $\varphi$  is in a proof from  $\emptyset$ , write  $\vdash \varphi$ ;  $\varphi$  is a **theorem**.

$$\mathbf{FO-Theorems} = \{\varphi \mid \vdash \varphi\}$$

**Prop:** FO-Theorems  $\in$  r.e.

**Proof:**

1. **for**  $i := 1$  to  $\infty$  {
2.       **for** each string  $S$  of length  $i$  {
3.           **if** ( $S$  is a correct, FO proof)
4.           **then** output LastLine( $S$ )
5.       **}** }



# Example FO Theorem

**Prop:**  $\vdash (x = y \rightarrow y = x)$

**Proof:**

1.	$(x = y \wedge x = x) \rightarrow (x = x \rightarrow y = x)$	AX 1c
2.	$x = x$	AX 1a
3.	$x = x \rightarrow$ $((x = y \wedge x = x) \rightarrow (x = x \rightarrow y = x)) \rightarrow$ $(x = y \rightarrow y = x)$	AX 0
4.	$((x = y \wedge x = x) \rightarrow (x = x \rightarrow y = x)) \rightarrow$ $(x = y \rightarrow y = x)$	MP 2, 3
5.	$x = y \rightarrow y = x$	MP 1, 4

3:  $\alpha \rightarrow ((\beta \wedge \alpha) \rightarrow (\alpha \rightarrow \gamma)) \rightarrow (\beta \rightarrow \gamma)$



# Soundness Theorem

If  $\Gamma \vdash \varphi$  then  $\Gamma \models \varphi$

FO-THEOREMS  $\subseteq$  FO-VALID

**Proof:** The axioms are valid and M.P. preserves validity.  $\square$

**Cor: Proofs Preserve Truth:** If  $\Gamma \vdash \varphi$  Then  $\Gamma \models \varphi$ .

If  $\mathcal{A} \models \Gamma$  and  $\Gamma \vdash \varphi$  Then  $\mathcal{A} \models \varphi$ .

**Proof:** The axioms are valid and M.P. preserves truth, i.e.,  
If  $\Gamma \models \varphi$  and  $\Gamma \models \varphi \rightarrow \psi$  Then  $\Gamma \models \psi$ .

Thus, by induction on the length of a proof from  $\Gamma$ , every line is true in every model of  $\Gamma$ .  $\square$

# Towards The Completeness Theorem

**Hilbert:** Complete Axiomatization for First-Order logic?

**Gödel:** [in his 1929 Ph.D. thesis] Yes!

Lect. 14: will prove Papadimitriou's Axioms complete.

Equivalent systems, i.e., also sound and complete:

- Natural Deduction
- Resolution

To prove Papadimitriou complete, we next show that Papadimitriou is closed under several **Meta Theorems**.

# Papadimitriou is Closed under $\forall$ intro

If  $\Gamma \vdash \varphi$  and  $x$  does not occur freely in  $\Gamma$  Then  $\Gamma \vdash \forall x(\varphi)$

**Proof:** By induction on the length of the proof of  $\varphi$  from  $\Gamma$ .

**Base case:**  $n = 1$ ,  $\varphi \in \text{AXIOMS}$ :  
 $\forall x(\varphi) \in \text{AXIOMS}$ .

**Base case:**  $n = 1$ ,  $\varphi \in \Gamma$ .

$\varphi \rightarrow \forall x(\varphi)$	AX3 since $x$ not free in $\varphi$ .
$\varphi$	in $\Gamma$
$\forall x(\varphi)$	M.P.

**Inductive case:** Assume true for all proofs of length  $< n$ .

$\alpha_1, \alpha_2, \dots, \alpha_{n-1}, \varphi$  is a proof from  $\Gamma$ .

By induction,  $\Gamma \vdash \forall x(\alpha_i), 1 \leq i \leq n - 1$ .

$\varphi$  follows from  $\alpha_i, \alpha_j$  by M.P.,  $i, j < n$       $\alpha_j = (\alpha_i \rightarrow \varphi)$

1.      $\forall x(\alpha_i)$
2.      $\forall x(\alpha_i \rightarrow \varphi)$
3.      $\forall x((\alpha_i \rightarrow \varphi) \rightarrow ((\forall x)(\alpha_i) \rightarrow (\forall x)\varphi))$      AX4
4.      $\forall x(\alpha_i) \rightarrow \forall x(\varphi)$      M.P. 2,3
5.      $\forall x(\varphi)$      M.P. 1,4



# Papadimitriou is Closed under $\rightarrow$ intro

**If**  $\Gamma \cup \{\varphi\} \vdash \psi$  **Then**  $\Gamma \vdash \varphi \rightarrow \psi$

**Proof:** By induction on  $n$ , length of proof:  $\Gamma \cup \{\varphi\} \vdash \psi$ .

**Base case:**  $n = 1$ ,  $\varphi = \psi$ :  $\psi \rightarrow \psi$  AX0

**Base case:**  $n = 1$ ,  $\psi \in \Gamma \cup$  AXIOMS:

1.  $\psi$   $\psi \in \Gamma \cup$  AXIOMS
2.  $\psi \rightarrow (\varphi \rightarrow \psi)$  AX0
3.  $\varphi \rightarrow \psi$  M.P., 1,2

**Inductive case:** Assume true for all proofs of length  $< n$ .

$\alpha_1, \alpha_2, \dots, \alpha_{n-1}, \psi$  is a proof from  $\Gamma \cup \{\varphi\}$ .

By induction,  $\Gamma \vdash \varphi \rightarrow \alpha_i, \quad 1 \leq i \leq n - 1.$

$\psi$  follows from  $\alpha_i, \alpha_j$  by M.P.,  $i, j < n$   $\alpha_j = (\alpha_i \rightarrow \psi)$

1.  $\varphi \rightarrow \alpha_i$   $\Gamma \vdash \varphi \rightarrow \alpha_i$
2.  $\varphi \rightarrow (\alpha_i \rightarrow \psi)$   $\Gamma \vdash \varphi \rightarrow \alpha_j$
3.  $(\varphi \rightarrow \alpha_i) \rightarrow ((\varphi \rightarrow (\alpha_i \rightarrow \psi)) \rightarrow (\varphi \rightarrow \psi))$  AX0
4.  $(\varphi \rightarrow (\alpha_i \rightarrow \psi)) \rightarrow (\varphi \rightarrow \psi)$  M.P., 1,3
5.  $\varphi \rightarrow \psi$  M.P., 2, 4



# Papadimitriou Closed Under Proof by Contradiction

**If**  $\Gamma \cup \{\varphi\} \vdash \perp$  **Then**  $\Gamma \vdash \neg\varphi$  → intro

**Proof:** Suppose  $\Gamma \cup \{\varphi\} \vdash \perp$

by → intro,  $\Gamma \vdash (\varphi \rightarrow \perp)$

1.  $\varphi \rightarrow \perp$
2.  $(\varphi \rightarrow \perp) \rightarrow \neg\varphi$       AX0
3.  $\neg\varphi$       M.P. 1,2

□

**Exercise: If**  $\Gamma \cup \{\neg\varphi\} \vdash \perp$  **Then**  $\Gamma \vdash \varphi$

# Change Bound Variable Lemma

If  $y$  is substitutable for  $x$  and does not occur freely in  $\varphi$ ,

Then  $\vdash \forall x(\varphi) \leftrightarrow \forall y(\varphi[x \leftarrow y])$

**Proof:** Show  $\forall x(\varphi) \vdash \varphi[x \leftarrow y]$

- |    |  |                        |
|----|--|------------------------|
| 1. | $\forall x(\varphi) \vdash \forall x(\varphi)$                                     | Assumption             |
| 2. | $\forall x(\varphi) \vdash \forall x(\varphi) \rightarrow \varphi[x \leftarrow y]$ | AX2                    |
| 3. | $\forall x(\varphi) \vdash \varphi[x \leftarrow y]$                                | M.P., 1,2              |
| 4. | $\forall x(\varphi) \vdash \forall y(\varphi[x \leftarrow y])$                     | $\forall$ Intro, 3     |
| 5. | $\vdash \forall x(\varphi) \rightarrow \forall y(\varphi[x \leftarrow y])$         | $\rightarrow$ Intro, 4 |

The converse is similar.



# Add a New Constant MetaTheorem $\exists$ Elim

If  $\Gamma \vdash \exists x(\varphi)$  and  $\Gamma, \varphi[x \leftarrow c] \vdash \psi$  where  $c$  does not occur in  $\Gamma, \varphi$ , or  $\psi$ , Then  $\Gamma \vdash \psi$ .

1.  $\Gamma, \varphi[x \leftarrow c] \vdash \psi$
2.  $\Gamma \vdash \varphi[x \leftarrow c] \rightarrow \psi$   $\rightarrow$  Intro, 1
3.  $\Gamma \vdash \neg\psi \rightarrow \neg\varphi[x \leftarrow c]$  AX0, MP, 2
4.  $\Gamma, \neg\psi \vdash \neg\varphi[x \leftarrow c]$  MP, 3
5.  $\Gamma, \neg\psi \vdash \neg\varphi[x \leftarrow z]$   $z$  new, Lemma next slide, 4
6.  $\Gamma, \neg\psi \vdash \forall z(\neg\varphi[x \leftarrow z])$   $\forall$  Intro, 5
7.  $\Gamma, \neg\psi \vdash \forall x(\neg\varphi)$  Change Bound Variable, 6
8.  $\Gamma, \neg\psi \vdash \neg\forall x(\varphi)$  given
9.  $\Gamma, \neg\psi \vdash \perp$   $\perp$  intro, 7, 8
10.  $\Gamma \vdash \psi$  Proof by Contradiction, 9

# Replace New Constant by New Variable Lemma

**Lemma:** If  $\Delta \vdash \alpha[x \leftarrow c]$ ,  
where neither  $z$  nor  $c$  occurs in  $\Delta$  or  $\alpha$ ,  
**Then**  $\Delta \vdash \alpha[x \leftarrow z]$

**Proof:** By induction on length of proof of  $\Delta \vdash \alpha[x \leftarrow c]$

Since  $z$  and  $c$  are new, we can systematically replace  $c$  by  $z$ .

All instances of the axioms remain valid. □

**Prop:**  $\vdash \forall xyz((x = y \wedge y = z) \rightarrow x = z)$

1.	$\{(x = y \wedge y = z)\} \vdash x = y$	$\wedge$ Elim
2.	$\{(x = y \wedge y = z)\} \vdash y = z$	$\wedge$ Elim
3.	$\vdash x = y \rightarrow y = x$	slide 5
4.	$\{(x = y \wedge y = z)\} \vdash y = x$	MP 1, 3
5.	$\vdash z = z$	AX 1a
6.	$\{(x = y \wedge y = z)\} \vdash (y = x \wedge z = z)$	$\wedge$ Intro 4, 5
7.	$\vdash (y = x \wedge z = z) \rightarrow (y = z \rightarrow x = z)$	AX 1c
8.	$\{(x = y \wedge y = z)\} \vdash (y = z \rightarrow x = z)$	MP 6, 7
9.	$\{(x = y \wedge y = z)\} \vdash x = z$	MP 2, 8
10.	$\vdash (x = y \wedge y = z) \rightarrow x = z$	$\rightarrow$ Intro 9
11.	$\vdash \forall z((x = y \wedge y = z) \rightarrow x = z)$	$\forall$ Intro. 10
12.	$\vdash \forall y \forall z((x = y \wedge y = z) \rightarrow x = z)$	$\forall$ Intro. 11
13.	$\vdash \forall x \forall y \forall z((x = y \wedge y = z) \rightarrow x = z)$	$\forall$ Intro. 12