Lect 14, 2008: recall FO AXIOMS: all generalizations of:

AX0: Tautologies on at most three boolean variables

AX1a: $t = t$ for any term t

AX1b: $(t_1 = t'_1 \wedge \cdots \wedge t_k = t'_k) \rightarrow f(t_1, \ldots, t_k) = f(t'_1, \ldots, t'_k)$
for terms $t_1, \ldots, t'_k$ $f \in \Phi$, $r(f) = k$

Ax1c: $(t_1 = t'_1 \wedge \cdots \wedge t_k = t'_k) \rightarrow (R(t_1, \ldots, t_k) \rightarrow R(t'_1, \ldots, t'_k))$
for terms $t_1, \ldots, t'_k$, $R \in \Pi$, $r(R) = k$

AX2: $\forall x(\varphi) \rightarrow \varphi[x \leftarrow t]$ t substitutable for x in φ

AX3: $\varphi \rightarrow \forall x(\varphi)$ where x does not occur freely in φ

AX4: $\forall x(\varphi \rightarrow \psi) \;\rightarrow\; (\forall x(\varphi) \rightarrow \forall x(\psi))$

Soundness Theorem

If $\vdash \varphi$ then $\models \varphi$;   FO-THEOREMS $\subseteq$ FO-VALID

Cor: Proofs Preserve Truth: If $\Gamma \vdash \varphi$ Then $\Gamma \models \varphi$.

Meta Theorems: Papadimitriou closed under:

- $\forall$ intro
- $\rightarrow$ intro
- proof by contradiction
- $\exists$ elim

Plan For Today's Lecture:

We will prove the **completeness** of Papadimitriou's axiomatization of first-order logic. The goal is to prove that any valid sentence is a theorem: $\models \varphi \quad \Rightarrow \quad \vdash \varphi$.

History:

- Early in the 20th century, Hilbert stated his "Entscheidungsproblem": give a decision procedure for first-order validity.
- In his 1929 Ph.D. thesis, Gödel proved the Completeness Theorem for first-order logic.
- In 1949, Leon Henkin published a particularly clear proof of the completeness theorem and this is the one that I will present today.

Definitions:

Γ is **consistent** iff we can't prove a contradiction from it: $\Gamma \nvdash \bot$.

Γ is **satisfible** iff it has a model, $\mathcal{A} \models \Gamma$.

Completeness Theorem: If Γ is consistent then Γ is satisfiable.

Cor: If $\models \varphi$ Then $\vdash \varphi$

Proof: [Assuming the Completeness Th.] Let φ be valid.

Suppose that $\{\neg\varphi\}$ is consistent, i.e., $\{\neg\varphi\} \nvdash \bot$.

Therefore, by the Completeness Theorem, $\{\neg\varphi\}$ is true in some model, i.e., $\mathcal{A} \models \neg\varphi$.

But this is impossible because φ is valid, so $\mathcal{A} \models \varphi$.

$\bot$

Therefore, $\{\neg\varphi\} \vdash \bot$.

Therefore $\vdash \varphi$ $\square$

Henkin's Proof of the Completeness Theorem:

- Start with consistent $\Gamma_0 \subset \mathcal{L}(\Sigma)$
- Add new constant symbols: $c_1, c_2, \ldots$ forming Σ'
- Order all formulas $\varphi_1, \varphi_2, \ldots$ from $\mathcal{L}(\Sigma')$ so that c_i does not occur in $\varphi_1, \ldots \varphi_i$
- Inductively, have Γ_{i-1} is consistent. Conclude that so is, $\Gamma_i = \Gamma_{i-1} \cup \{\psi_i, \alpha_i\}$ where $\psi_i \in \{\varphi_i, \neg\varphi_i\}$ and if $\psi_i = \exists v(\gamma(v))$ then $\alpha_i = \gamma(c_i)$.

Inductively, have Γ_{i-1} is consistent.

Conclude so is, $\Gamma_i = \Gamma_{i-1} \cup \{\psi_i, \alpha_i\}$
where $\psi_i \in \{\varphi_i, \neg\varphi_i\}$ and if $\psi_i = \exists v(\gamma(v))$ then $\alpha_i = \gamma(c_i)$.

Proof:

- If $\Gamma_{i-1} \cup \varphi_i \vdash \bot$ and $\Gamma_{i-1} \cup \neg\varphi_i \vdash \bot$

  Then $\Gamma_{i-1} \vdash \bot$.

  $\bot$ Thus, $\Gamma_{i-1} \cup \{\psi_i\}$ is consistent

  Note non-computable step: do we add φ_i or $\neg\varphi_i$?

- Assume $\psi_i = \exists v(\gamma(v))$

  If $\Gamma_{i-1} \cup \{\exists v(\gamma(v)), \gamma(c_i)\} \vdash \bot$

  then by $\exists$ elim, $\Gamma_{i-1} \cup \{\exists v(\gamma(v)))\} \vdash \bot$

$\square$

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- Let $\Delta = \bigcup_{i=0}^{\infty} \Gamma_i$. Conclude that Δ is **consistent**.

  Proof: $\Gamma_0 \subseteq \Gamma_1 \subseteq \Gamma_2 \subseteq \cdots \subseteq \Delta$

  Suppose $\Delta \vdash \bot$

  Since proofs are finite, for some $i, \Gamma_i \vdash \bot$.

  $\bot$ $\square$

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- Let $\Delta \;=\; \bigcup_{i=0}^{\infty} \Gamma_i$. Conclude that Δ is **consistent**.
- Δ is **complete**, i.e., for all $\varphi \in \mathcal{L}(\Sigma')$, $\Delta \vdash \varphi$, or $\Delta \vdash \neg\varphi$
- Δ has **Henkin property**: if $\Delta \vdash \exists v(\gamma(v))$, then for some constant symbol, c, $\Delta \vdash \gamma(c)$
- Use Δ to build model $\mathcal{A}$ satisfying Δ, and thus also Γ_0.

Use Δ to build model $\mathcal{A}$ satisfying Δ

- Since Δ is complete, it knows the answer to every question, i.e., it tells us all of $\mathcal{A}$'s properties.
- Since Δ satisfies the Henkin Property, if Δ thinks that $\exists x(\gamma(x))$, then there is a witness, c_i such that $\Delta \vdash \gamma(c_i)$.
- Thus, the witness constants, $c_1, c_2, \ldots$, witness all properties that hold!
- Thus, it might **seem** appropriate to let the universe, $|\mathcal{A}|$, of $\mathcal{A}$ be the set of witness constant symbols: $\{c_1, c_2, \ldots\}$.
- However, it could be the case, e.g., that $\Delta \vdash c_2 = c_7$. In this case, we can't have c_2 and c_7 be different members of $|\mathcal{A}|$. We must choose a single **representative** from among the constants that Δ thinks are equal to c_2 to be in the universe.

Definition: $c_i \equiv c_j$ **iff** $\Delta \vdash c_i = c_j$

Claim: $\equiv$ is an equivalence relation.

Proof: $\vdash c_i = c_i \qquad \vdash c_i = c_j \to c_j = c_i$

$\vdash (c_i = c_j \wedge c_j = c_k) \to c_i = c_k \qquad \square$

$$|\mathcal{A}| = \{[c_i] \mid i \in \mathbf{N}\}$$

$$x^{\mathcal{A}} = [c_n], \quad \text{s.t. } \Delta \vdash x = c_n$$

$$R^{\mathcal{A}} = \{\langle [c_{i_1}], \ldots, [c_{i_{r(R)}}] \rangle \mid \Delta \vdash R(c_{i_1}, \ldots, c_{i_{r(R)}})\}$$

$$f^{\mathcal{A}} = \{\langle [c_{i_1}], \ldots, [c_{i_{r(f)+1}}] \rangle \mid \Delta \vdash f(c_{i_1}, \ldots, c_{i_{r(f)}}) = c_{i_{r(f)+1}}\}$$

Claim: $f^{\mathcal{A}}$ and $R^{\mathcal{A}}$ are well defined, i.e., the definitions do not depend on which representative of the equivalence class $[c_i]$ we choose.

Proof: Because $\mathcal{A} \models$ AX1b and AX1c. $\square$

Claim: $\mathcal{A} \models \Delta$.

Proof: For each $\varphi \in \Delta$, we show by induction on φ that $\mathcal{A} \models \varphi$.
See Papadimtriou, p. 110 for the details. $\square$

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- Let $\Delta = \bigcup_{i=0}^{\infty} \Gamma_i$. Conclude that Δ is **consistent**.
- Δ is **complete**, i.e., for all $\varphi \in \mathcal{L}(\Sigma')$, $\Delta \vdash \varphi$, or $\Delta \vdash \neg\varphi$
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- Use Δ to build model $\mathcal{A}$ satisfying Δ, and thus also Γ_0.

Compactness Theorem:

Let Γ be a set of first-order formulas.

Suppose every finite subset of Γ has a model.

Then Γ has a model.

Proof: Suppose that Γ is inconsistent, i.e., $\Gamma \vdash \bot$

Then since proofs are finite, there is a finite subset $\Gamma_0 \subseteq \Gamma$ s.t. $\Gamma_0 \vdash \bot$.

Since Γ_0 is finite, it has a model, $\mathcal{A} \models \Gamma_0$

Thus, by the Soundness Theorem, $\mathcal{A} \models \bot$

$\Rightarrow\Leftarrow$ Thus Γ is consistent.

By the Completeness Theorem, Γ has a model. $\Box$

Compactness Applications

$$\begin{aligned}\text{Theory}(\mathbf{N}) &= \{\varphi \in \mathcal{L}(\Sigma_N) \mid \mathbf{N} \models \varphi\} \\ \Gamma &= \text{Theory}(\mathbf{N}) \cup \{c > 0, c > 1, c > 2, c > 3, \ldots\}\end{aligned}$$

Claim: Γ has a model, $N' \models \Gamma$

Proof: Every finite subset of Γ is satisfiable by $(\mathbf{N}, i)$ for i sufficiently large. Thus, by Compactness, Γ is satisfiable. ☐

Thus, there is a countable model,

$$N' \models \text{Theory}(\mathbf{N}); \qquad N' \not\cong \mathbf{N}$$

Thus, $\mathcal{L}(\Sigma_N)$ cannot uniquely characterize $\mathbf{N}$.

“Connectedness” is not expressible in $\mathcal{L}(\Sigma_g)$

Proof: Suppose that $\chi \equiv$ “I am connected.”

$$\Gamma \quad = \quad \{\chi\} \cup \left\{\neg\text{dist}(s,t,1), \neg\text{dist}(s,t,2), \neg\text{dist}(s,t,3), \ldots\right\}$$

$$\text{dist}(x_0, x_n, n) \quad \equiv \quad \exists x_1 \cdots x_{n-1} \left(\bigwedge_{i=0}^{n-1} (x_i = x_{i+1} \lor E(x_i, x_{i+1})) \right)$$

Every finite subset of Γ is satisfiable.
By Compactness, Γ has a model, $\mathcal{A}$.
$\mathcal{A}$ is not connected but $\mathcal{A} \models \chi$. $\Rightarrow\Leftarrow$ $\square$

Thus “Connectedness” is not expressible in the first-order language of graphs.

Downward Lowenheim-Skolem Theorem:

If a set of first-order formulas, Γ, has any model at all, Then it has a **countable** model.

Proof: Suppose that Γ has a model.

By the Soundness Theorem, Γ is consistent.

By our proof of the Completeness Theorem, there is a model, $\mathcal{A}$ of Γ whose universe consists of the equivalence classes of the witness constants: $c_1, c_2, \ldots$, i.e., $\mathcal{A}$ is countable. $\square$

The set of real numbers is **uncountable**. But if we define a first-order vocabulary to talk about $\mathbf{R}$, we get a first-order theory, $\mathrm{Theory}(\mathbf{R})$, the set of sentences that are true of $\mathbf{R}$. This theory has a countable model!

Kind of weird.

Zermelo Fraenkel Set Theory plus the Axiom of Choice

ZFC is a first-order axiomatization of set theory.

Extensionality: $\forall x, y(\forall z(z \in x \leftrightarrow z \in y) \rightarrow x = y)$

If two sets have the same elements then they are equal

Empty Set: $\exists x \forall y(y \notin x)$

There is a (unique) empty set; let's call it $\emptyset$

Plus 7 more axioms . . .

> "The reader should be reasonably convinced even on a first reading that the axioms easily encompass all of traditional mathematics,"
>
> Paul J. Cohen, *Set Theory and the Continuum Hypothesis*

ZFC is something like what Hilbert was asking for.

Back to the Compactness Theorem

If ZFC is consistent, then it has a countable model, $\mathcal{A}$.

Thus, if you believe Paul Cohen, then $\mathcal{A}$ is a model of all of mathematics. In particular, it has $\mathbf{N}, \wp(\mathbf{N}), \wp(\wp(\mathbf{N})), \ldots$.

Furthermore, ZFC $\vdash$ "$\wp(\mathbf{N})$ is uncountable."

Thus, $\mathcal{A} \models$ "$\wp(\mathbf{N})$ is uncountable."

How can this be, when all these sets are part of the countable universe, $|\mathcal{A}|$?

What does $\mathcal{A} \models$ "$\wp(\mathbf{N})$ is uncountable" mean?

It means that there is no $f : \mathbf{N} \overset{\text{1:1}}{\underset{\text{onto}}{\to}} \wp(\mathbf{N})$.

Now $\mathbf{N}^{\mathcal{A}}, \wp(\mathbf{N})^{\mathcal{A}} \subseteq |A|$, so they are both countable, so there is such an f.

However, $f \not\in |\mathcal{A}|$

$\mathcal{A} \models$ "$\wp(\mathbf{N})$ is uncountable"