601 Lect 16: Recall From Last Time

Th: Every primitive recursive function is representable by a bounded formula.

Cor: K is representable by a bounded formula.

Th: FO-THEOREMS is **r.e.** complete.

Gödel's Incompleteness Th: There is no r.e. set of sentences Γ such that

- $\mathbf{N} \models \Gamma$, and
- $\Gamma \vdash \text{Theory}(\mathbf{N})$.

Recall Some Complexity Classes

Def: A set $A \subseteq \Sigma^\star$ is in $\mathbf{DTIME}[t(n)]$ iff there exists a deterministic, multi-tape TM, M, and a constant c, such that,

1. $A \quad = \quad \mathcal{L}(M) \quad \equiv \quad \left\{w \in \Sigma^\star \;\middle|\; M(w) = 1\right\}$, and
2. $\forall w \in \Sigma^\star$, $M(w)$ halts within $c \cdot t(|w|)$ steps.

Def: A set $A \subseteq \Sigma^\star$ is in $\mathbf{DSPACE}[s(n)]$ iff there exists a deterministic, multi-tape TM, M, and a constant c, such that,

1. $A \quad = \quad \mathcal{L}(M)$, and
2. $\forall w \in \Sigma^\star$, $M(w)$ uses at most $c \cdot s(|w|)$ work-tape cells.

(The input tape is considered “read-only” and not counted as space used.)

Thm: For any functions $t(n) \geq n,\ s(n) \geq \log n$,

$$\mathbf{DTIME}[t(n)] \quad \subseteq \quad \mathbf{DSPACE}[t(n)]$$

$$\mathbf{DSPACE}[s(n)] \quad \subseteq \quad \mathbf{DTIME}[2^{O(s(n))}]$$

Proof: Let M be a $\mathbf{DSPACE}[s(n)]$ TM, $w \in \Sigma_0^\star$, $n = |w|$

$M(w)$ has k tapes and uses at most $cs(n)$ work-tape cells.

$M(w)$ has at most $2^{k' s(n)}$ possible configurations:

$$|Q| \quad \cdot \quad (n + cs(n) + 2)^k \quad \cdot \quad |\Sigma|^{cs(n)} \quad < \ 2^{k' s(n)}$$

states · # head positions · # tape contents

Thus, after $2^{k' s(n)}$ steps, $M(w)$ must be in an infinite loop. □

$$\mathbf{NTIME}[t(n)] \equiv \text{problems accepted by NTMs in time } t(n)$$

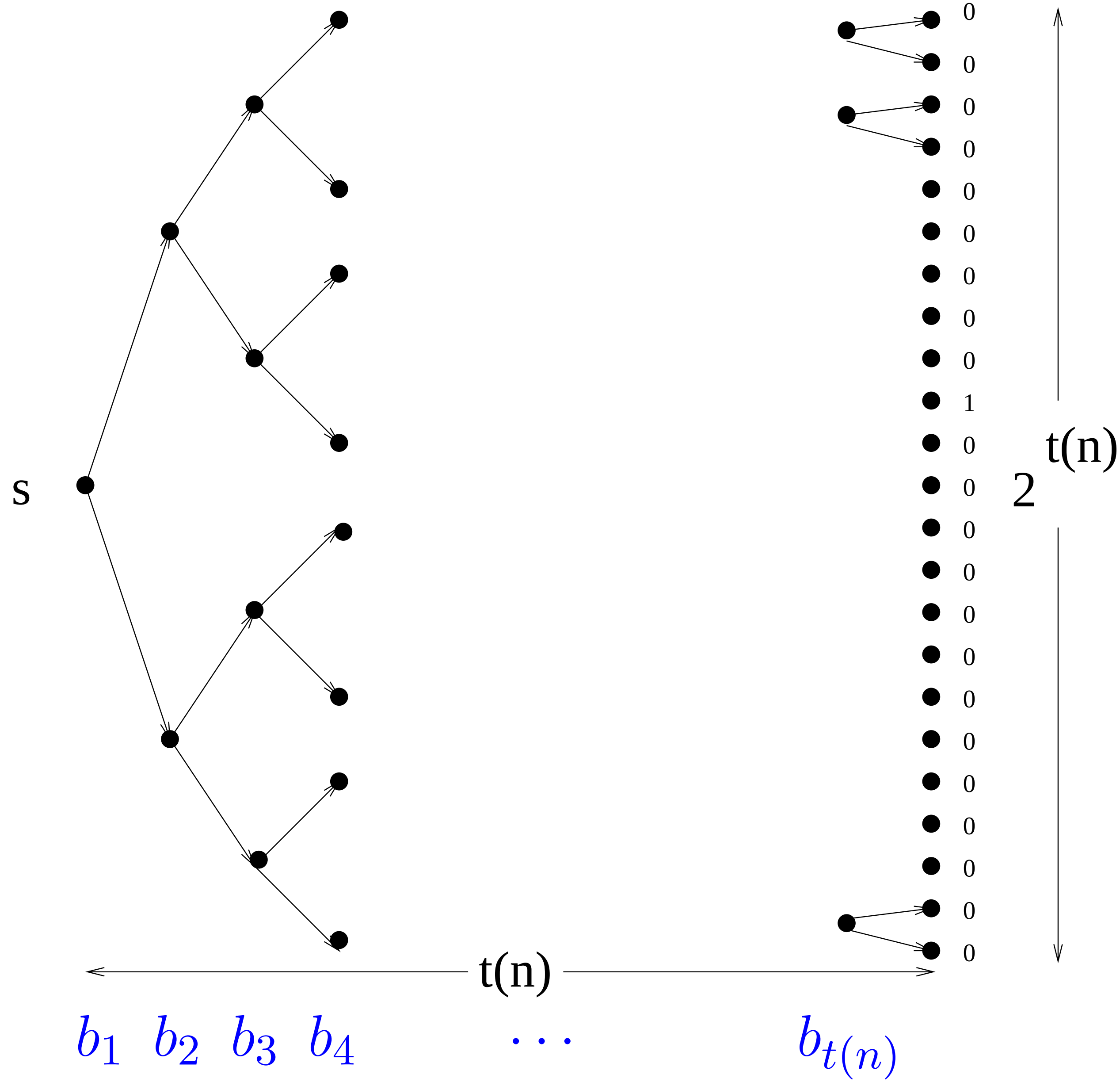
$$\mathbf{NP} \equiv \mathbf{NTIME}[n^{O(1)}] \equiv \bigcup_{i=1}^{\infty} \mathbf{NTIME}[n^i]$$

Theorem For any function $t(n)$,

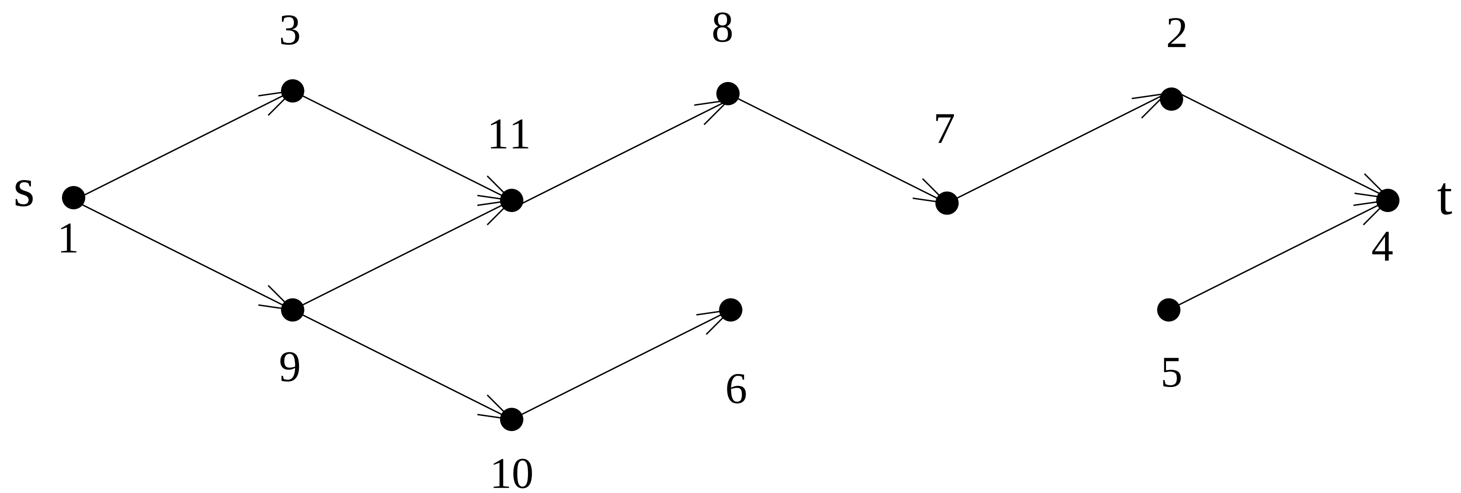
$$\mathbf{DTIME}[t(n)] \subseteq \mathbf{NTIME}[t(n)] \subseteq \mathbf{DSPACE}[t(n)] \subseteq \mathbf{DTIME}[2^{O(t(}$$

Cor: $\mathbf{L} \subseteq \mathbf{P} \subseteq \mathbf{NP} \subseteq \mathbf{PSPACE}$

NSPACE$[s(n)]$ is the set of problems accepted by NTMs using at most $O(s(n))$ space on each branch.



Def: $\text{REACH} = \{G \mid s \xrightarrow{\star} t\}$



Prop: $\text{REACH} \in$ **NL** $=$ **NSPACE**$[\log n]$

$b := s$
for $c := 1$ **to** $n = |V|$ **do {**
  if $b = t$ **then accept**
  $a := b$
  choose new b
  if $(\neg E(a, b))$ **then reject }**
reject

a: 2

b: 4

accept!

Def: Problem T is **complete** for complexity class $\mathcal{C}$ iff

1. $T \in \mathcal{C}$, and
2. $\forall A \in \mathcal{C}\,(A \leq T)$

Reductions now must be in $F(\mathbf{L})$.

Thm: REACH is complete for **NL**.
Proof: Let $A \in$ **NL**, $A = \mathcal{L}(N)$, uses $c \log n$ bits of worktape.
Input w, $\quad n = |w|$

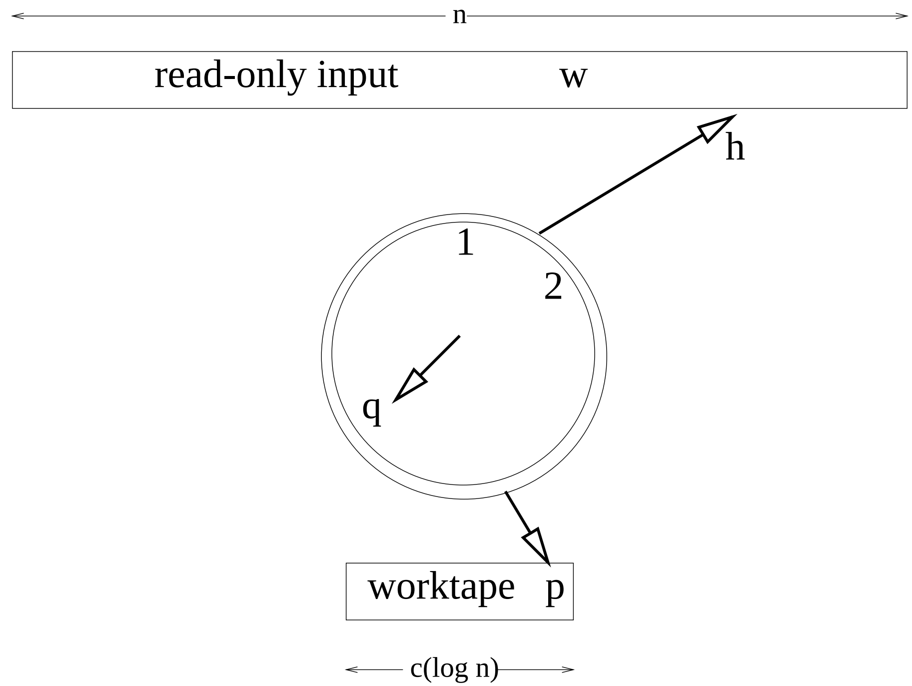
$$w \mapsto \mathrm{CompGraph}(N, w) \quad = \quad (V, E, s, t)$$

$$V \quad = \quad \left\{ \mathrm{ID} = \langle q, h, p \rangle \;\middle|\; q \in \text{States}(N), h \leq n, |p| \leq c\lceil \log n \rceil \right\}$$

$$E \quad = \quad \left\{ (\mathrm{ID}_1, \mathrm{ID}_2) \;\middle|\; \mathrm{ID}_1(w) \xrightarrow[N]{} \mathrm{ID}_2(w) \right\}$$

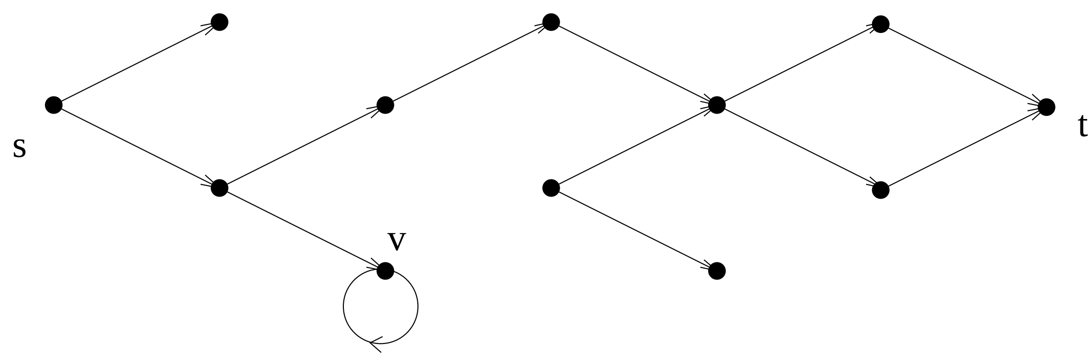
$$s \quad = \quad \text{initial ID}$$

$$t \quad = \quad \text{accepting ID}$$



$$
\begin{aligned}
w &\mapsto \text{CompGraph}(N, w) \quad = \quad (V, E, s, t) \\
V &= \left\{ \text{ID} = \langle q, h, p \rangle \;\middle|\; q \in \mathsf{States}(N), h \le n, |p| \le c\lceil \log n \rceil \right\} \\
E &= \left\{ (\text{ID}_1, \text{ID}_2) \;\middle|\; \text{ID}_1(w) \underset{N}{\longrightarrow} \text{ID}_2(w) \right\} \\
s &= \text{initial ID} \\
t &= \text{accepting ID}
\end{aligned}
$$

Claim: $w \in \mathcal{L}(N) \ \Leftrightarrow\ \text{CompGraph}(N, w) \in \text{REACH}$



$\square$

Cor: $\mathbf{NL} \subseteq \mathbf{P}$

Proof: $\text{REACH} \in \mathbf{P}$

$\mathbf{P}$ is closed under (logspace) reductions.

i.e., $(B \in \mathbf{P} \quad \wedge \quad A \leq B) \quad \Rightarrow \quad A \in \mathbf{P}$ $\square$

Hierarchy Theorems

If $f(n)$ is a $\mathcal{C}$-constructible function;

$\mathcal{C}$ is DSPACE, NSPACE, DTIME, or NTIME; and,

if $g(n)$ is sufficiently smaller than $f(n)$

Then $\mathcal{C}[g(n)]$ is strictly contained in $\mathcal{C}[f(n)]$.

$g(n)$ **sufficiently smaller** than $f(n)$ means:

$$\lim_{n\to\infty}\left(\frac{g(n)}{f(n)}\right) = 0 \qquad\qquad \lim_{n\to\infty}\left(\frac{g(n)\log(g(n))}{f(n)}\right) = 0$$

$\mathcal{C} =$ DSPACE, NSPACE, NTIME $\qquad\qquad$ $\mathcal{C} =$ DTIME

Definition:

Function $f : \mathbf{N} \rightarrow \mathbf{N}$ is **$\mathcal{C}$-constructible** if the map

$$1^n \mapsto f(n)$$

is computable in the complexity class $\mathcal{C}[f(n)]$.

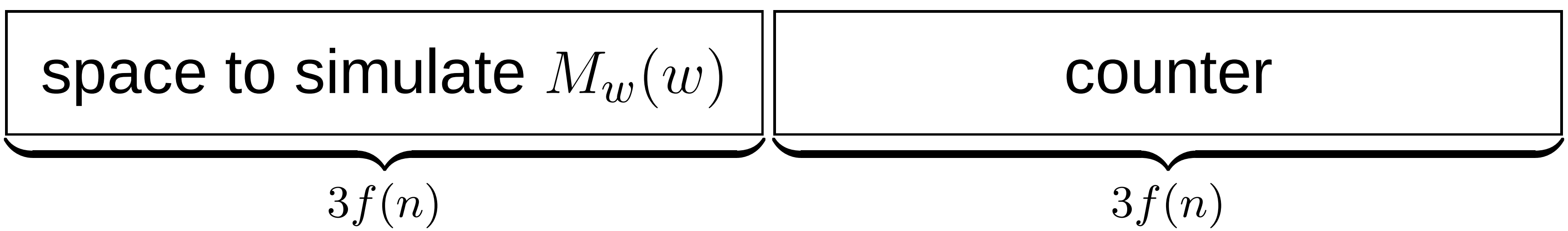
For example a function $f(n)$ is DSPACE-constructible if the function $f(n)$ can be deterministically computed from the input 1^n, using space at most $O[f(n)]$.

Fact: All reasonable functions greater than or equal to $\log n$ are DSPACE-constructible, and all reasonable functions greater than or equal to n are DTIME-constructible.

Space Hierarchy Thm: If $f \geq \log n$ is space constructible and $\lim_{n \to \infty} \left(\frac{g(n)}{f(n)} \right) = 0$, Then $\mathbf{DSPACE}[g(n)] \subsetneq \mathbf{DSPACE}[f(n)]$.

Proof: Build $\mathbf{DSPACE}[f(n)]$ machine, D, on input: w, $n = |w|$

1. Mark off $6f(n)$ tape cells, (f space constructible)
2. Simulate $M_w(w)$ using space $3f(n)$, time $\leq 2^{3f(n)}$
3. **if** ($M_w(w)$ needs more space or time) **then accept**
4. **else if** ($M_w(w)$ **= accept)** **then reject**
5. **else accept** // ($M_w(w)$ **= reject)**



Claim: $\mathcal{L}(D) \in$ **DSPACE**$[f(n)] -$ **DSPACE**$[g(n)]$

Proof: $\mathcal{L}(D) \in$ **DSPACE**$[f(n)]$ by construction.

Suppose $\mathcal{L}(D) \in$ **DSPACE**$[g(n)]$.

Let $\mathcal{L}(M_w) = \mathcal{L}(D)$, M_w uses $cg(n)$ space.

Choose N s.t. $\forall n > N \; (cg(n) < f(n))$.

Choose w', $M_{w'}(\cdot) = M_w(\cdot)$, $|w'| > N$

On input w', D successfully simulates $M_{w'}(w')$ in $3f(n)$ space and $2^{3f(n)}$ time.

$$w' \in \mathcal{L}(D) \;\Leftrightarrow\; w' \notin \mathcal{L}(M_{w'}) \;\Leftrightarrow\; w' \notin \mathcal{L}(M_w) \;\Leftrightarrow\; w' \notin \mathcal{L}(D)$$

$$\Rightarrow\Leftarrow$$

$\square$

$F(\mathcal{C})$

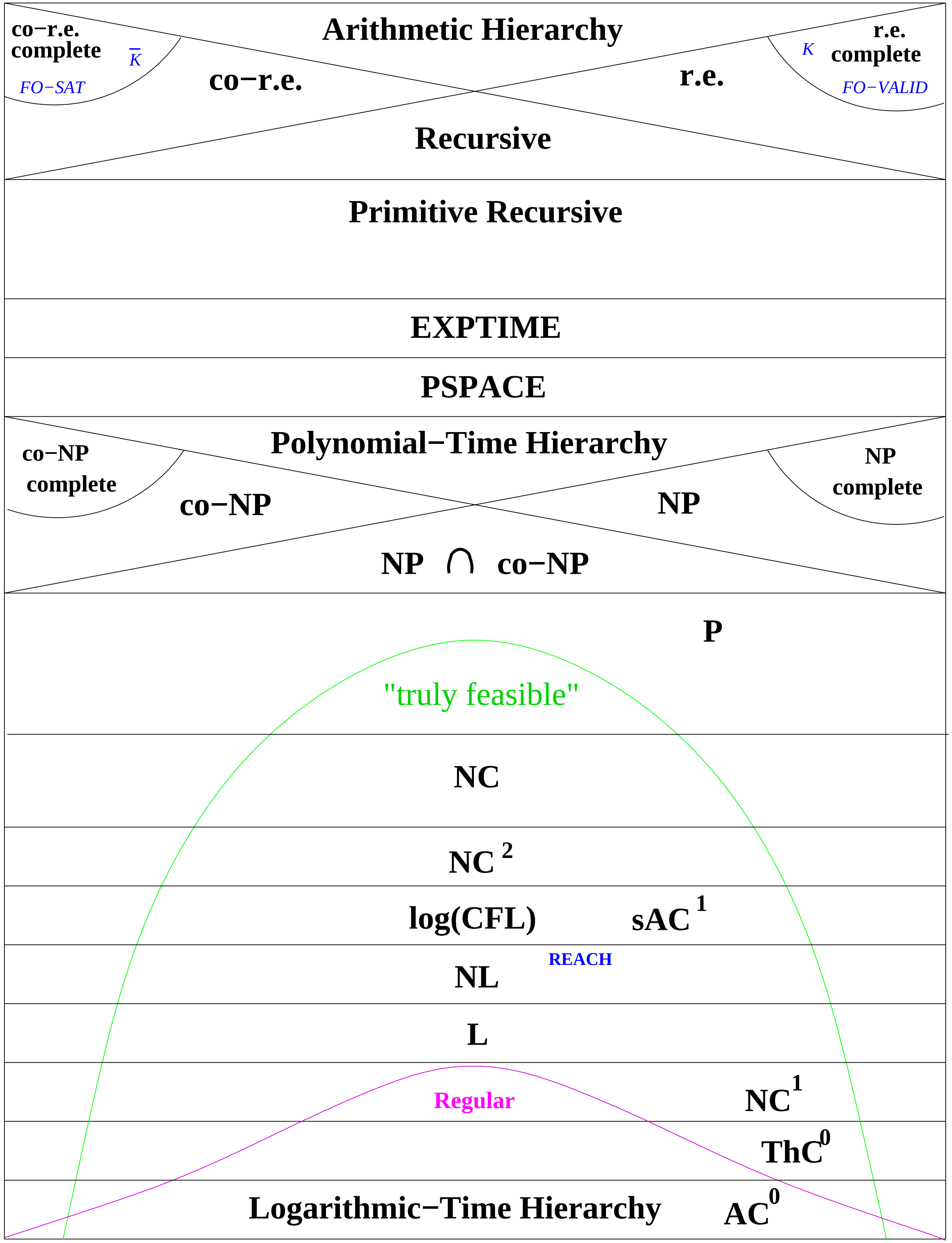
For any complexity class $\mathcal{C}$, define $F(\mathcal{C})$, the total, polynomially-bounded functions computable in $\mathcal{C}$ as follows:

$$F(\mathcal{C}) \;=\; \left\{ h : \Sigma^\star \to \Sigma^\star \;\middle|\; \begin{array}{l} \exists k \, \forall x \, (|h(x)| \leq k|x|^k \\ \text{and} \quad \text{bit-graph}(h) \in \mathcal{C}) \end{array} \right\}$$

$$\text{bit-graph}(h) \;=\; \{ \langle x, i, b \rangle \mid \text{bit } i \text{ of } h(x) \text{ is } b \}$$

Idea: $f \in F(\mathcal{C})$ iff

1. f is polynomially bounded, and,
2. bit i of $f(w)$ is uniformly computable in $\mathcal{C}$ and co-$\mathcal{C}$.



Arithmetic Hierarchy
co−r.e. complete
$\overline{K}$
FO−SAT
co−r.e.
r.e.
K
r.e. complete
FO−VALID
Recursive
Primitive Recursive
EXPTIME
PSPACE
Polynomial−Time Hierarchy
co−NP complete
co−NP
NP
NP complete
NP ∩ co−NP
P
"truly feasible"
NC
NC^2
log(CFL)
sAC^1
REACH
NL
L
Regular
NC^1
ThC^0
Logarithmic−Time Hierarchy
AC^0