Recall From Last Time

S is **reducible** to T, $S \le T$, iff $\exists$ total, recursive $f : \mathbf{N} \to \mathbf{N}$,

$$\forall w \in \mathbf{N}(w \in S \quad \Leftrightarrow \quad f(w) \in T)$$

[In the future we will insist that $f \in F(\mathbf{L})$.]

Th: Suppose $S \le T$. Then,

1. If T is **r.e.**, then S is **r.e.**.
2. If T is co-**r.e.**, then S is co-**r.e.**.
3. If T is **Recursive**, then S is **Recursive**.

C is r.e.-*complete* iff 1. $C \in$ **r.e.** and 2. $\forall A \in$ **r.e.** $(A \le C)$

Th: K, HALT, and $A_{0,17}$ are r.e. complete.

Dichotomy

We will see very often that natural problems are complete for natural complexity classes.

Today, we will see that most decision problems about regular sets or context-free languages are in $\mathbf{P}$, but some such problems are r.e.-complete!

Today's Main Theorem

The following problem is co-**r.e.**-complete:

$$\Sigma^\star\mathrm{CFL} \quad = \quad \{G \mid G \text{ is a CFG; } \mathcal{L}(G) = \Sigma_G^\star\}$$

Proof: [Juris Hartmanis]

$\overline{\Sigma^\star\mathrm{CFL}} \in$ **r.e.** :

Input: G, let $\Sigma_G^\star = \{w_0, w_1, w_2, \ldots\}$

1. **for** $i := 0$ to ∞ {
2. **if** $w_i \notin \mathcal{L}(G)$, **then return**(1)}

[CFL-Member $= \{G, w \mid G$ a CFL, $w \in \mathcal{L}(G)\} \in$ **P**]

Prop: EMPTY is co-r.e. complete, where,

$$\text{EMPTY} \quad = \quad \{n \mid W_n = \emptyset\}$$

Proof: Exercise □

Claim: EMPTY $\le \Sigma^\star$CFL.

Cor: $\Sigma^\star$CFL is co-r.e. complete and thus not recursive.

Wanted: $g : \mathbf{N} \to \{0,1\}^\star$, s.t.

$$n \in \text{EMPTY} \quad \Leftrightarrow \quad g(n) \in \Sigma^\star\text{CFL}$$

$$\forall x (M_n(x) \neq 1) \quad \Leftrightarrow \quad \mathcal{L}(g(n)) = \Sigma_n^\star$$

$$M_n \text{ has no accepting computations} \quad \Leftrightarrow \quad \mathcal{L}(g(n)) = \Sigma_n^\star$$

Build g, s.t. $(M_n$ has no accepting comp$) \Leftrightarrow (\mathcal{L}(g(n)) = \Sigma_n^\star)$

Encode ID's of TM M_n conveniently for CFG $g(n)$:

M_n has alphabet $\{0, 1\}$, states $\{\hat{0}, \hat{1}, \ldots, \hat{q}\}$
$\hat{0} =$ halting state; $\hat{1} =$ start state

$$\mathrm{ID}_0 \quad = \quad \hat{1} \triangleright w_1\, w_2\, \cdots\, w_r \sqcup$$

Suppose M_n in state $\hat{1}$ looking at a "$\triangleright$" writes a "$\triangleright$" changes to state $\hat{3}$, and moves to the right.

$$\mathrm{ID}_1 \quad = \quad \triangleright\, \hat{3}\, w_1\, w_2\, \cdots\, w_r \sqcup$$

$\mathrm{YesComp}(n)$ =

$$\left\{ \mathrm{ID}_0 \# \mathrm{ID}_1^R \# \mathrm{ID}_2 \# \mathrm{ID}_3^R \# \cdots \# \mathrm{ID}_t \;\middle|\; \mathrm{ID}_0 \cdots \mathrm{ID}_t \ \begin{array}{c}\text{accepting} \\ \text{comp of } M_n\end{array} \right\}$$

Lemma: For each n, $\overline{\mathrm{YesComp}(n)}$ is a CFL.

Furthermore, $\exists g \in F(\mathbf{L})\ \forall n \in \mathbf{N}\ (\mathcal{L}(g(n)) = \overline{\mathrm{YesComp}(n)})$.

$$\Sigma_n \ = \ \{0, 1, \triangleright, \sqcup, \#, \hat{0}, \hat{1}, \ldots, \hat{q_n}\} \qquad [M_n \text{ has } q_n \text{ states}]$$

$$n \in \text{EMPTY} \quad \Leftrightarrow \quad \overline{\mathrm{YesComp}(n)} = \Sigma_n^\star \quad \Leftrightarrow \quad g(n) \in \Sigma^\star\mathrm{CFL}$$

Thus, g : EMPTY $\leq \Sigma^\star\mathrm{CFL}$

$$\overline{\text{YesComp}(n)} = U(n) \cup A(n) \cup D(n) \cup Z(n)$$

$$U(n) = \left\{w \in \Sigma_n^\star \;\middle|\; w \text{ not in form } \mathrm{ID}_0\# \cdots \#\mathrm{ID}_t\right\}$$

$$A(n) = \left\{w \in \Sigma_n^\star \;\middle|\; w \text{ doesn't start with initial ID of } M_n\right\}$$

$$D(n) = \left\{w \in \Sigma_n^\star \;\middle|\; \exists i(\mathrm{ID}_{i+1} \text{ doesn't follow from } \mathrm{ID}_i)\right\}$$

$$Z(n) = \left\{w \in \Sigma_n^\star \;\middle|\; w \text{ doesn't end with } \hat{0} \,\triangleright\, 1 \sqcup\right\}$$

Thus, g : EMPTY $\leq \Sigma^{\star}\mathrm{CFL}$

$$n \in \text{EMPTY} \iff \overline{\mathrm{YesComp}(n)} = \Sigma_n^{\star}$$
$$\iff g(n) \in \Sigma^{\star}\mathrm{CFL}$$

This completes the proof of the Claim.

It follows, as desired, that $\Sigma^{\star}$CFL is co-r.e. complete. □

Th: The following problems are in polynomial time.

$$
\begin{aligned}
\text{EmptyNFA} &= \{N \mid N \text{ is an NFA; } \mathcal{L}(N) = \emptyset\} \\
\Sigma^\star\text{DFA} &= \{D \mid D \text{ is a DFA; } \mathcal{L}(D) = \Sigma^\star\} \\
\text{MemberNFA} &= \{\langle N, w\rangle \mid N \text{ is an NFA; } w \in \mathcal{L}(N)\} \\
\text{EqualDFA} &= \{\langle D_1, D_2\rangle \mid D_1, D_2 \text{ DFAs; } \mathcal{L}(D_1) = \mathcal{L}(D_2)\} \\
\text{EmptyCFL} &= \{G \mid G \text{ is a CFG; } \mathcal{L}(G) = \emptyset\} \\
\text{MemberCFL} &= \{\langle G, w\rangle \mid G \text{ is a CFG; } w \in \mathcal{L}(G)\}
\end{aligned}
$$

EmptyNFA *Exercise*

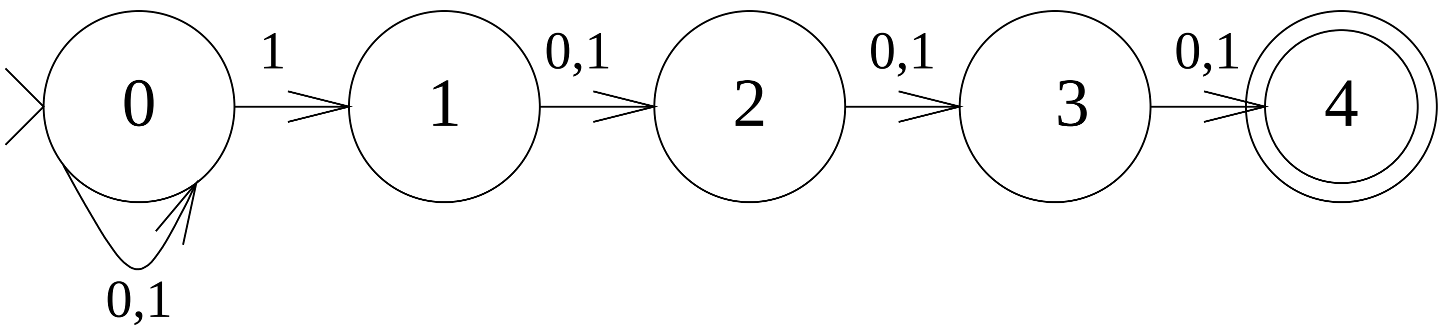
$$\Sigma^\star\text{DFA} = \{D \mid D \text{ is a DFA; } \mathcal{L}(D) = \Sigma^\star\}$$

$$D \in \Sigma^\star\text{DFA} \Leftrightarrow \overline{D} \in \text{EmptyNFA}$$

$$\text{MemberNFA} = \{\langle N, w\rangle \mid N \text{ is an NFA; } w \in \mathcal{L}(N)\}$$

1 1 0 0 1

$\{0\}$ $\{0,1\}$ $\{0,1,2\}$ $\{0,2,3\}$ $\{0,3,4\}$ $\{0,1,4\}$



EqualDFA $\quad Hw4$

EmptyCFL $\quad Hw2$

$$\text{MemberCFL} \quad = \quad \{\langle G, w\rangle \mid G \text{ is a CFG}; \; w \in \mathcal{L}(G)\}$$

CYK Dynamic Programming Algorithm:

1. Assume G in **Chomsky Normal Form:** $N \to AB$, $N \to a$.
2. **Input:** $w = w_1 w_2 \ldots w_n$; G, nonterminals $S, A, B, \ldots$
3. $N_{ij} \equiv \begin{cases} 1 & \text{if } N \xrightarrow{\star} w_i \cdots w_j \\ 0 & \text{otherwise} \end{cases}$
4. **return**(S_{1n})

$$N_{i,i} \;=\; \textbf{if } (\text{“}N \to w_i\text{”} \in R) \textbf{ then } 1 \textbf{ else } 0$$

$$N_{i,j} \;=\; \bigvee_{\text{“}N \to AB\text{”} \in R} \exists k (i \le k < j \;\wedge\; A_{i,k} \;\wedge\; B_{k+1,j})$$

Th: The following problems are in polynomial time.

$$\begin{aligned}
\text{EmptyNFA} &= \{N \mid N \text{ is an NFA; } \mathcal{L}(N) = \emptyset\} \\
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\text{EqualDFA} &= \{\langle D_1, D_2\rangle \mid D_1, D_2 \text{ DFAs; } \mathcal{L}(D_1) = \mathcal{L}(D_2)\} \\
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\text{MemberCFL} &= \{\langle G, w\rangle \mid G \text{ is a CFG; } w \in \mathcal{L}(G)\}
\end{aligned}$$

Th: The following problem is co-**r.e.**-complete:

$$\Sigma^\star\text{CFL} = \{G \mid G \text{ is a CFG; } \mathcal{L}(G) = \Sigma_G^\star\}$$

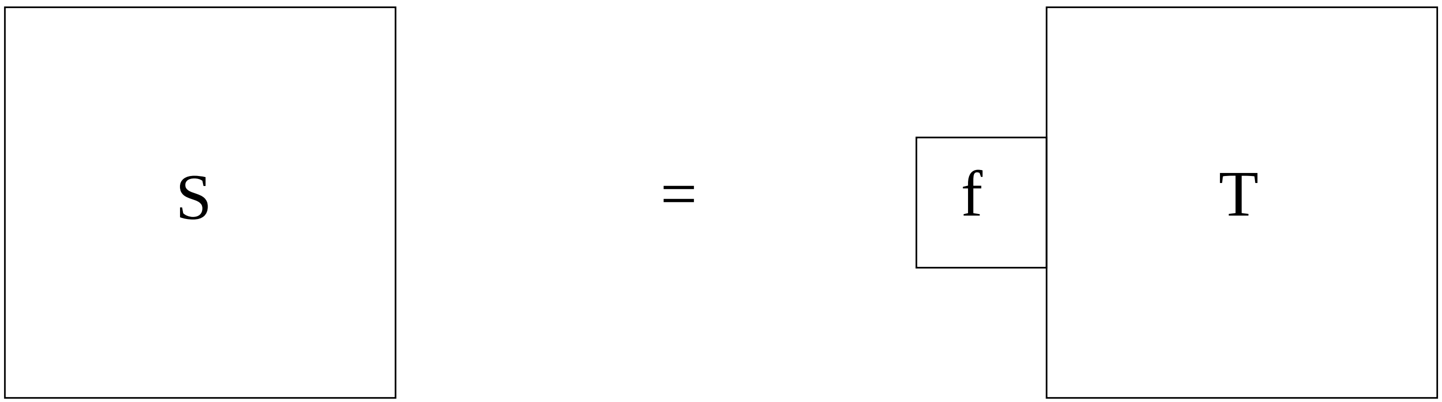
Intuition Concerning Reductions

S is *reducible* to T, $S \leq T$, iff $\exists f \in F(\,\mathbf{L})$,

$$\forall w \in \mathbf{N}(w \in S \quad \Leftrightarrow \quad f(w) \in T)$$

Intuition: $S \leq T$ iff the placement of a **very simple front end** f before a T-recognizer creates an S-recognizer.

$$\chi_S \quad = \quad \chi_T \circ f, \qquad \text{i.e.,} \quad \forall x(\chi_S(x) = \chi_T(f(x)))$$

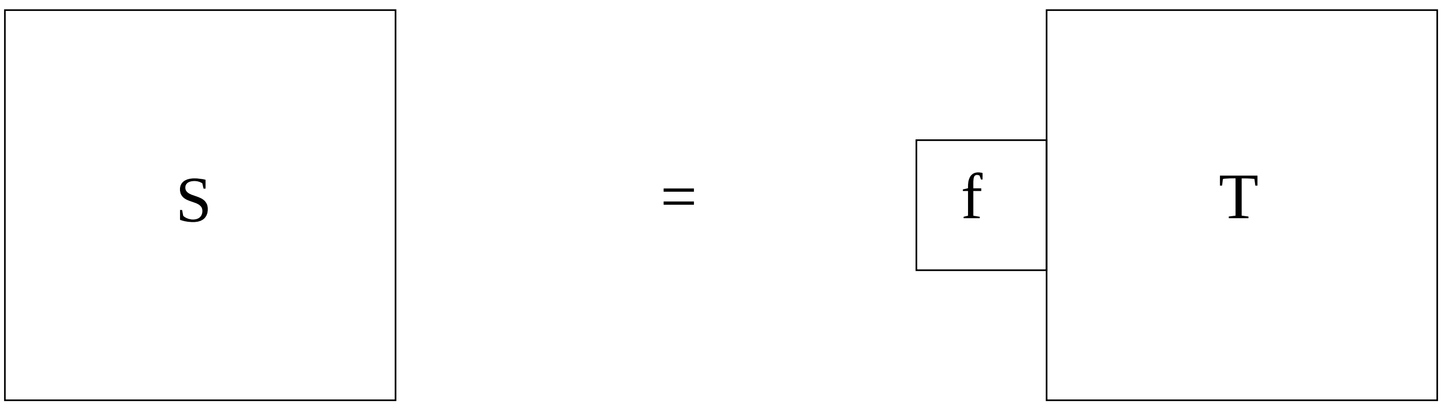


The Reduction Game

To build a reduction, f, from S to T you must solve the following puzzle:

"For each input, w, what membership question, $f(w)$, can I ask T such that the answer is the membership question $w \in ?S$."

$$\chi_S \quad = \quad \chi_T \circ f, \qquad \text{i.e.,} \quad \forall x(\chi_S(x) = \chi_T(f(x)))$$



Prop: $\mathrm{HALT} = \{P(n,m) \mid M_n(m) \text{ halts}\}$ is r.e.-complete.

Proof: Already seen HALT is r.e.. Must show $K \leq \mathrm{HALT}$.

Wanted: total, recursive f s.t. $\forall w (w \in K \;\Leftrightarrow\; f(w) \in \mathrm{HALT})$

$$M_w(w) = 1 \quad \Leftrightarrow \quad M_{L(f(w))}(R(f(w))) \text{ halts}$$

$$M_w(w) = 1 \quad \Leftrightarrow \quad M_\ell(r) \text{ halts}, \quad \text{where } f(w) = P(\ell, r)$$

Given w, let, $M_{\ell(w)} \; =$

Erase input	Write w	M_w	if 1 then halt else diverge

Letting $f(w) = P(\ell(w), 0)$, we have that

$$M_w(w) = 1 \quad \Leftrightarrow \quad M_{\ell(w)}(0) \text{ halts} \quad \Leftrightarrow \quad f(w) \in \mathrm{HALT} \quad \Box$$