601 Lecture 22: PSPACE

$$\mathbf{PSPACE} \quad = \quad \mathbf{DSPACE}[n^{O(1)}] \quad = \quad \mathbf{NSPACE}[n^{O(1)}]$$

- **PSPACE** consists of what we could compute with a feasible amount of hardware, but with no time limit.
- **PSPACE** is a large and very robust complexity class.
- With polynomially many bits of memory, we can search any implicitly-defined graph of exponential size. This leads to complete problems such as reachability on exponentially-large graphs.
- We can search the game tree of any board game whose configurations are describable with polynomially-many bits. This leads to complete problems concerning winning strategies.

PSPACE-Complete Problems

Recall from Lecture 20: **PSPACE** $=$ **ATIME**$[n^{O(1)}]$

Recall QSAT, the quantified satisfiability problem.

Prop: QSAT is **PSPACE**-complete.

Proof: QSAT $\in$ **ATIME**$[n]$ $\subseteq$ **PSPACE** (Lecture 20).

QSAT is hard for $\mathbf{ATIME}[n^k]$:

Let M be an $\mathbf{ATIME}[n^k]$ TM, w an input, $n = |w|$

Let M write down its n^k alternating choices, $c_1\ c_2\ \ldots\ c_{n^k}$.

Deterministic TM D evaluates the answer, i.e., for all inputs w, $M(w) = 1 \;\Leftrightarrow\; \exists c_1 \forall c_2 \cdots \exists c_{n^k} \,(D(\bar{c}, w) = 1)$

By Cook's Theorem $\exists$ reduction $f : \mathcal{L}(D) \leq \text{SAT}$:

$$D(\bar{c}, w) = 1 \quad \Leftrightarrow \quad f(\bar{c}, w) \in \text{SAT}$$

Let the new boolean variables in $f(\bar{c}, w)$ be $d_1 \ldots d_{t(n)}$.

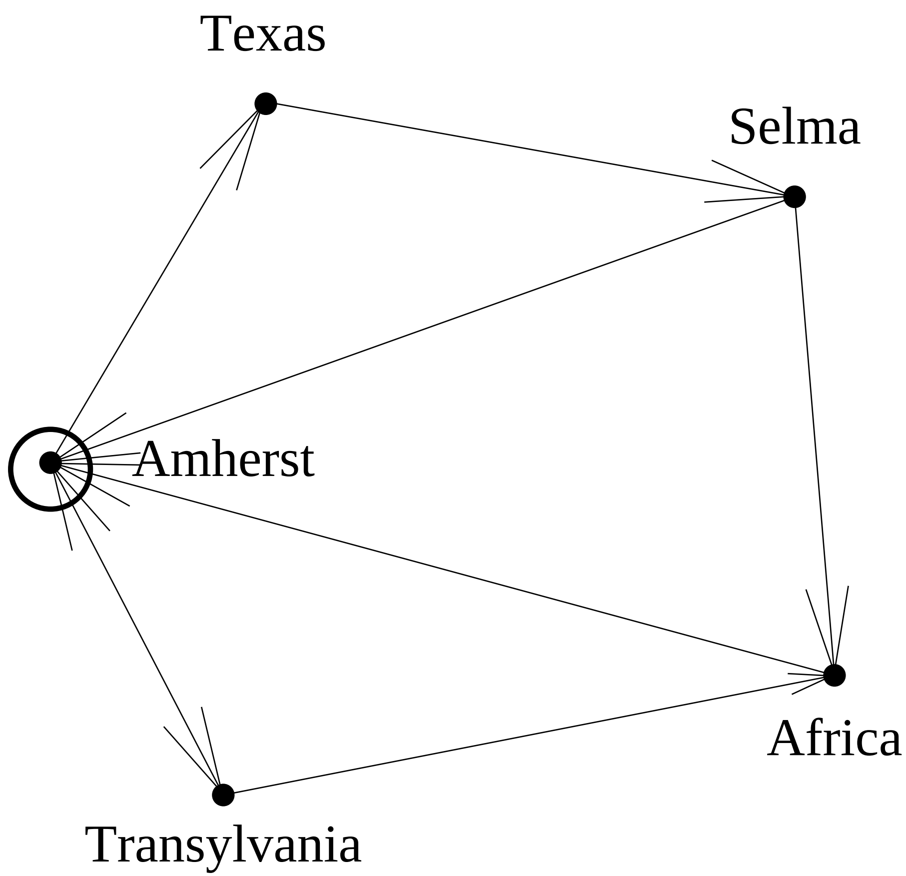
$M(w) = 1 \;\Leftrightarrow\;$ "$\exists c_1 \forall c_2 \cdots \exists c_{n^k}\, d_1 \ldots d_{t(n)}\,(f(\bar{c}, w))$" $\in$ QSAT

□

Geography is a two-person game.

1. E chooses a vertex v_1 with an edge from s.
2. A chooses v_2, having an edge from v_1
3. E chooses v_3, have an edge from v_2, etc.

No vertex may be chosen twice. Whoever moves last wins.



Let GEOGRAPHY be the set of positions in geography games s.t. $\exists$ has a winning strategy.

Prop: GEOGRAPHY is **PSPACE**-complete.

Proof:

GEOGRAPHY $\in$ **PSPACE:**

search the polynomial-depth game tree. A polynomial-size stack suffices.

Show: QSAT $\leq$ GEOGRAPHY

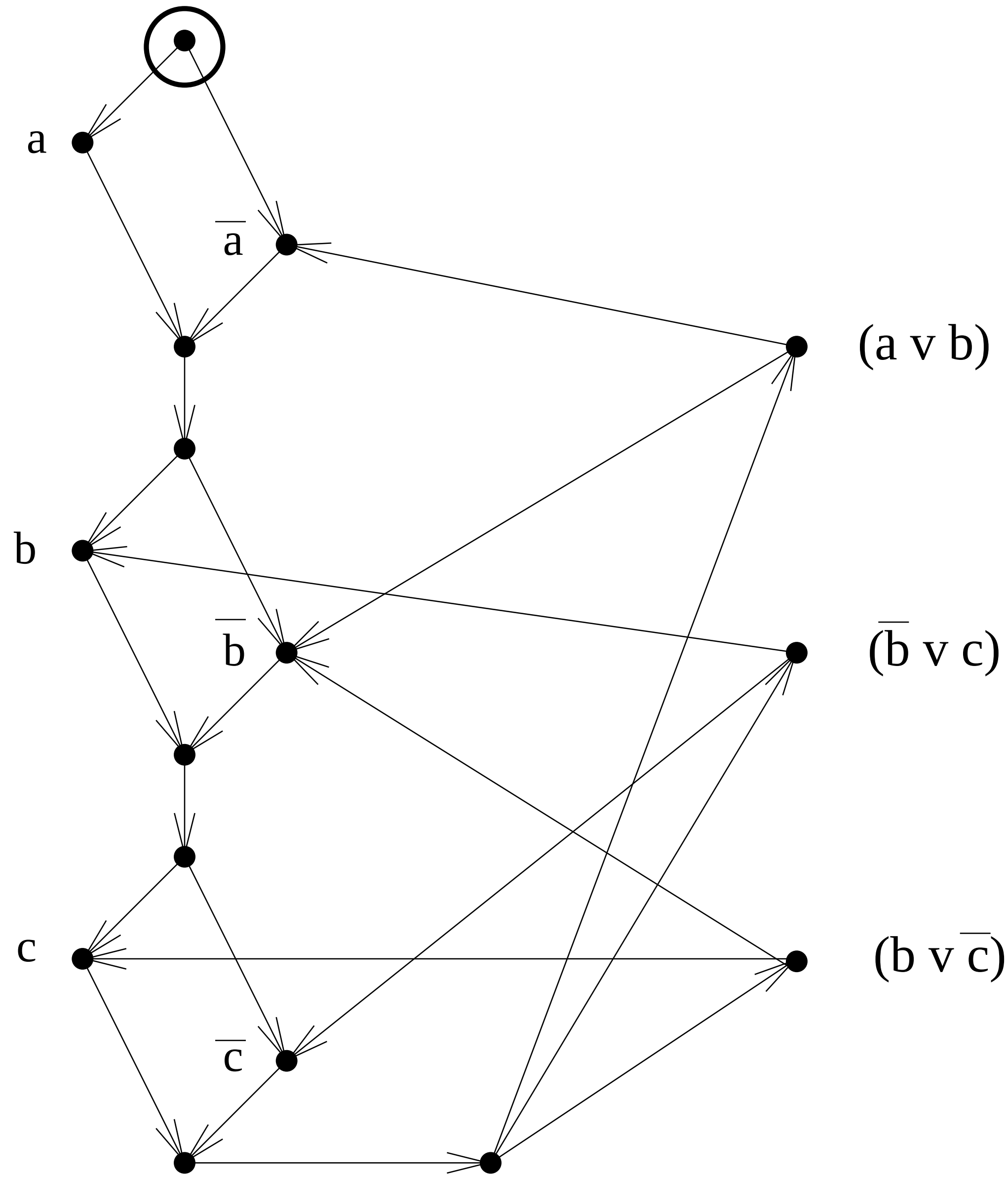
Given formula, φ, build graph G_φ s.t. $\exists$ chooses existential variables; $\forall$ chooses universal variables.

$$\varphi \equiv \exists a \forall b \exists c$$

$$[(a \vee b) \wedge$$

$$(\bar{b} \vee c) \wedge$$

$$(b \vee \bar{c})]$$



□

Def: A **succinct** representation of a graph is

$$G(n, C, s, t) \;=\; (V, E, s, t)$$

where C is a boolean circuit with $2n$ inputs and

$$V \;=\; \left\{ w \;\middle|\; w \in \{0,1\}^n \right\}$$

$$E \;=\; \left\{ (w, w') \;\middle|\; C(w, w') = 1 \right\}$$

$$\text{SUCCINCT REACH} = \left\{ (n, C, s, t) \;\middle|\; G(n, C, s, t) \in \text{REACH} \right\}$$

Prop: SUCCINCT REACH $\in$ **PSPACE**

Why?

Prop: SUCCINCT REACH $\in$ **PSPACE**

Why?

Remember Savitch's Thm:

$$\text{REACH} \in \mathbf{NSPACE}[\log n] \subseteq \mathbf{DSPACE}[(\log n)^2]$$

$$\text{SUCCINCT REACH} \in \mathbf{NSPACE}[n] \subseteq \mathbf{DSPACE}[n^2] \subseteq \mathbf{PSPACE}$$

$\square$

Prop: SUCCINCT REACH is **PSPACE**-complete.

Proof: Let M be a **DSPACE**$[n^k]$ TM, input w, $n = |w|$

$$M(w) = 1 \quad \leftrightarrow \quad \text{CompGraph}(M, w) \in \text{REACH}$$

$$\text{CompGraph}(n, w) \quad = \quad (V, E, s, t)$$

$$V \quad = \quad \{\text{ID} = \langle q, h, p \rangle \mid q \in \text{States}(N), h \leq n, |p| \leq c\, n^k\}$$

$$E \quad = \quad \{(\text{ID}_1, \text{ID}_2) \mid \text{ID}_1(w) \xrightarrow[M]{} \text{ID}_2(w)\}$$

$$s \quad = \quad \text{initial ID}$$

$$t \quad = \quad \text{accepting ID}$$

□

Succinct Representation of $\mathrm{CompGraph}(n, w)$:

$$V \;=\; \left\{\mathrm{ID} = \langle q, h, p\rangle \;\middle|\; q \in \text{States}(N), h \le n, |p| \le c\, n^k\right\}$$

$$E \;=\; \left\{(\mathrm{ID}_1, \mathrm{ID}_2) \;\middle|\; \mathrm{ID}_1(w) \underset{M}{\longrightarrow} \mathrm{ID}_2(w)\right\}$$

Let $V = \{0,1\}^{c' n^k}$

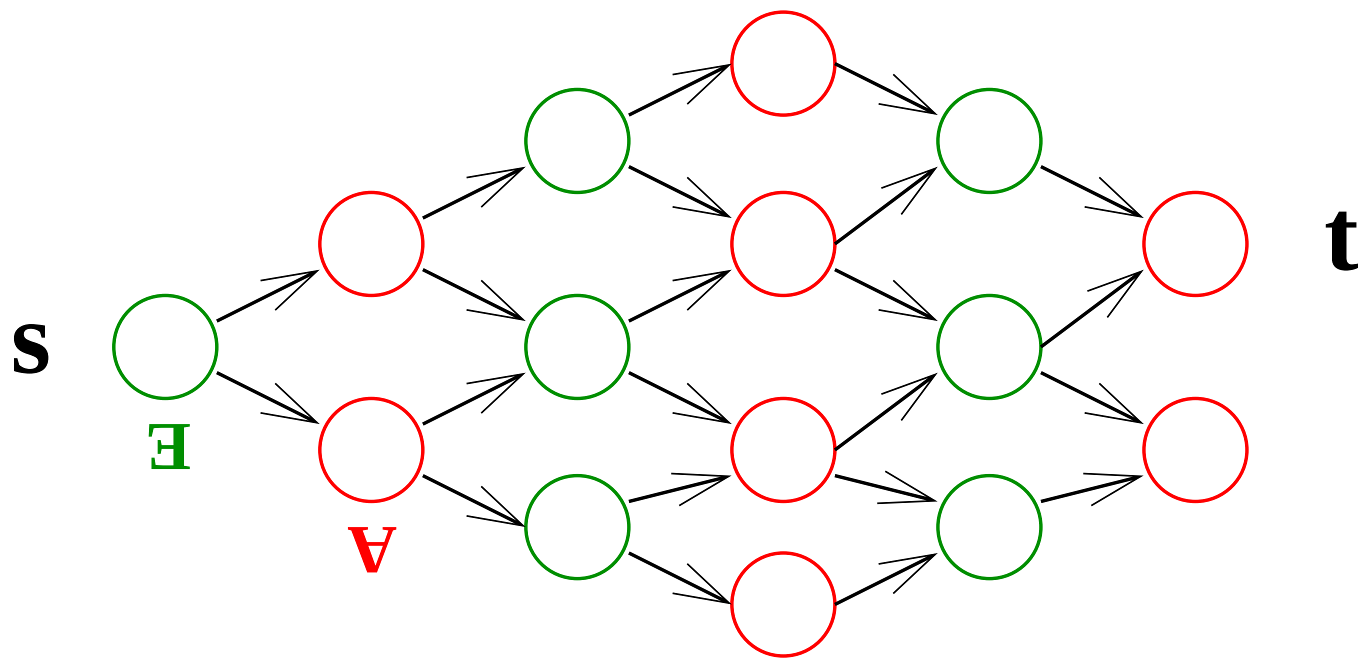
Build circuit C_w: on input $u, v \in V$, accept iff $u \underset{M}{\longrightarrow} v$.

$$M(w) = 1 \quad \Leftrightarrow \quad G(c'n^k, C_w, s, t) \in \text{SUCCINCT REACH} \quad \Box$$

The vertices of an **alternating graph**, $G = (V, A, E)$, are split into: **existential vertices** and **universal vertices**.

Def: vertex t is **reachable** from vertex s in G iff

1. $s = t$, or
2. s is existential and for some edge, $\langle s, a\rangle \in E$, t is reachable from a, or,
3. s is universal and there is an edge leaving s and for all edges, $\langle s, a\rangle \in E$, t is reachable from a.



Def: Let

$$\text{AREACH} = \{G = (V, A, E, s, t) \mid t \text{ is reachable from } s\}$$

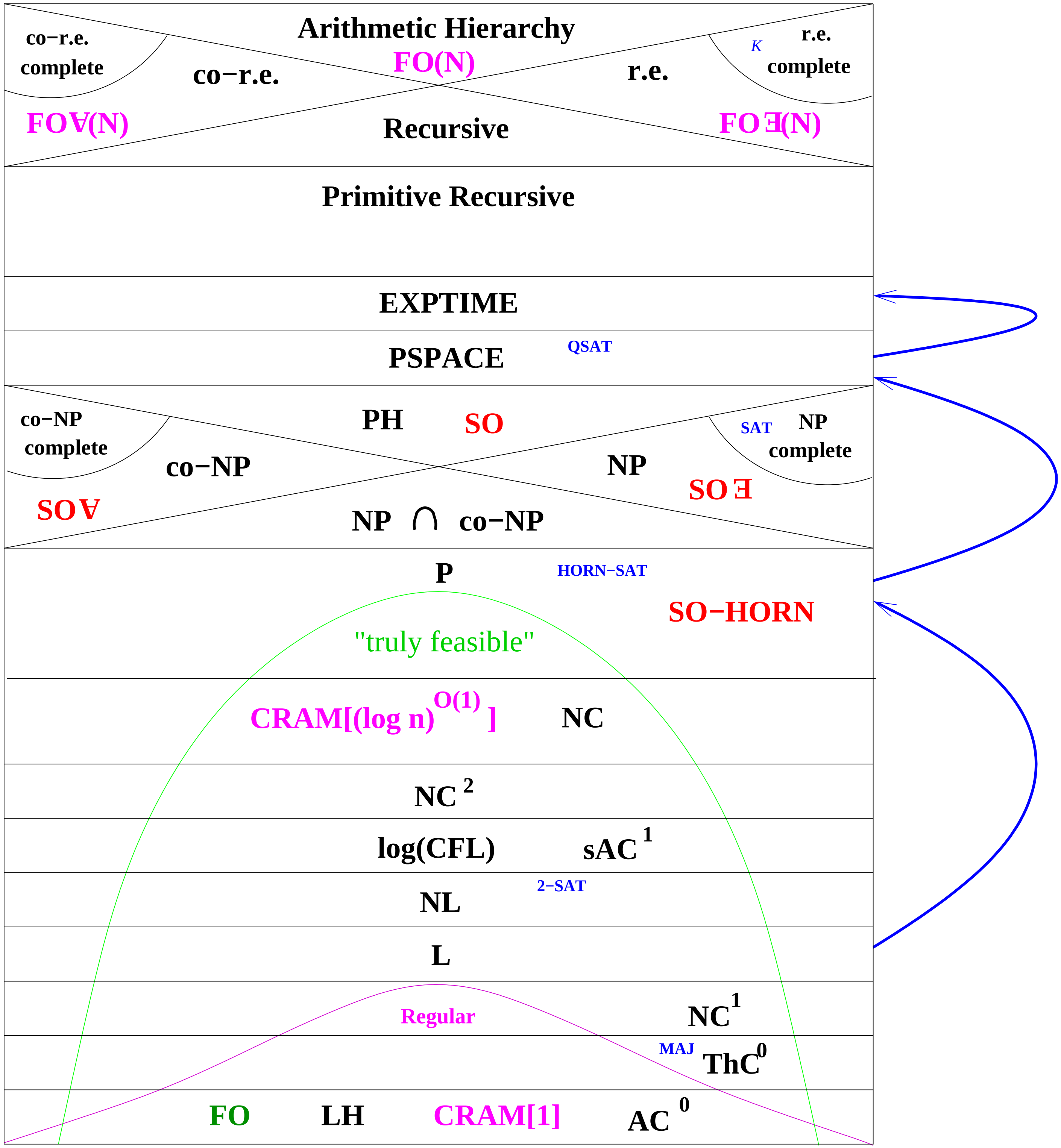
Prop: AREACH is **P** complete.

Proof: HW 10.

□

Cor: CVP and MCVP are **P** complete.

Proof: In HW10 we showed that AREACH $\leq$ MCVP and the identity map is a reduction from MCVP to CVP. □


Arithmetic Hierarchy
FO(N)
co−r.e. complete
co−r.e.
r.e.
K
r.e. complete
FO∀(N)
Recursive
FO∃(N)
Primitive Recursive
EXPTIME
PSPACE
QSAT
PH
SO
co−NP complete
co−NP
NP
SAT
NP complete
SO∃
SO∀
NP ∩ co−NP
P
HORN−SAT
SO−HORN
"truly feasible"
CRAM[(log n)^O(1)]
NC
NC 2
log(CFL)
sAC 1
NL
2−SAT
L
Regular
NC1
MAJ
ThC0
FO
LH
CRAM[1]
AC 0