Comp Sci 691M:	Notes and Assignment	Sept. 9, 1996

Notes and Comments: Today we completed talking about the syntax and semantics of Propositional Logic including the definition of valuation, truth tables, tautologies, and the notion of semantic implication: $\Gamma \models \varphi$ is read, "Γ semantically implies φ," and means that every valuation that assigns "1" to all the formulas in Γ also assigns "1" to φ.

We showed that $\{\wedge, \neg\}$ and $\{\vee, \neg\}$ are functionally complete sets and that every formula in PROP is equivalent to a formula in conjunctive normal form (CNF) and to one in disjunctive normal form (DNF). Note that we did this as suggested in Exercises 7, page 27.

Reading: Please read van Dalen, pp. 21 – 39.

Exercises: Please continue to read all the exercises. On pages 27 – 28, you should do exercises 4, 11, 14.

Comments and hints on previous exercises: Ex. 4., p. 13: "Show that if φ occurs in a shortest formation sequence of ψ then φ is a subformula of ψ." This is a little bit subtle. You should first do exercise 3. which tells us that every subformula of ψ occurs in every formation sequence of ψ. Now, you should prove by induction on ψ that the set of formulas occurring in a shortest formation sequence for ψ is **exactly** the set of subformulas of ψ.

Ex. 12., p. 14: The problem asks you to do some ground work and actually prove Theorem 1.1.6 (Definition by Recursion). This is perhaps a bit too subtle, so do not bother to do it.

If you tried it and want to know the answer, then read on. We are given some set A, and the mappings: $H_\square : A^2 \to A$, for each binary connective $\square \in \{\wedge, \vee, \to, \leftrightarrow\}$, the map $H_\neg : A \to A$, as well as the map $H_{at} : \{\bot, p_i \,|\, i \in \mathbf{N}\} \to A$. These define the function $F : \mathrm{PROP} \to A$ as follows,

$$F(\varphi) \;=\; \begin{cases} H_{at}(\varphi) & \text{if } \varphi \text{ is atomic} \\ H_\square(F(\alpha), F(\beta)) & \text{if } \varphi = (\alpha \square \beta) \\ H_\neg(F(\alpha)) & \text{if } \varphi = (\neg \alpha) \end{cases}$$

Exercise 12 asks you to formalize this a bit more. Don't bother, but here is how you would get started, basically defining F the way we defined PROP: Let $F^\star$ be the **smallest subset** of $\mathrm{PROP} \times A$ such that:

1. If α is atomic, Then $\langle \alpha, H_{at}(\alpha) \rangle \in F^\star$
2. If $\alpha = (\varphi \square \psi)$ and there exist pairs $\langle \varphi, a \rangle, \langle \psi, b \rangle \in F^\star$, Then $\langle \alpha, H_\square(a, b) \rangle \in F^\star$.
3. If $\alpha = (\neg \varphi)$ and there exists a pair $\langle \varphi, a \rangle \in F^\star$, Then $\langle \alpha, H_\neg(a) \rangle \in F^\star$.