FO Isomorphism Theorems and Descriptive Complexity

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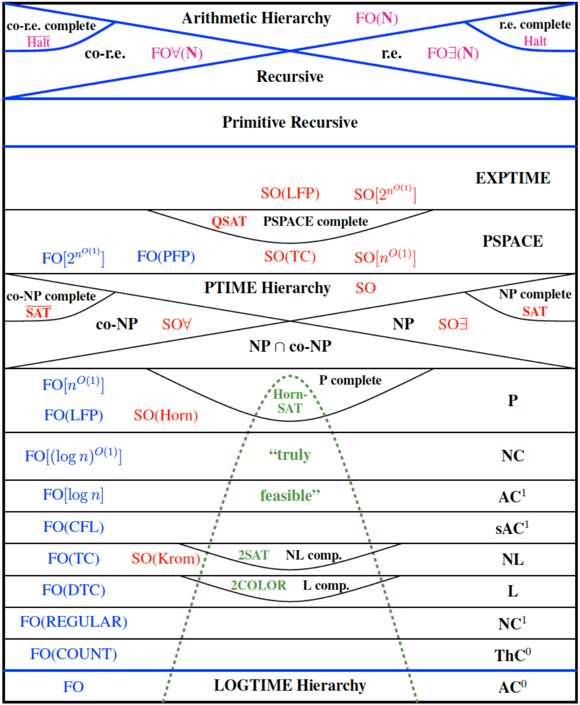
“truly feasible” is the informal set of problems we can solve exactly on all reasonably sized instances.



$$P = \bigcup_{k=1}^{\infty} \mathrm{DTIME}[n^k]$$

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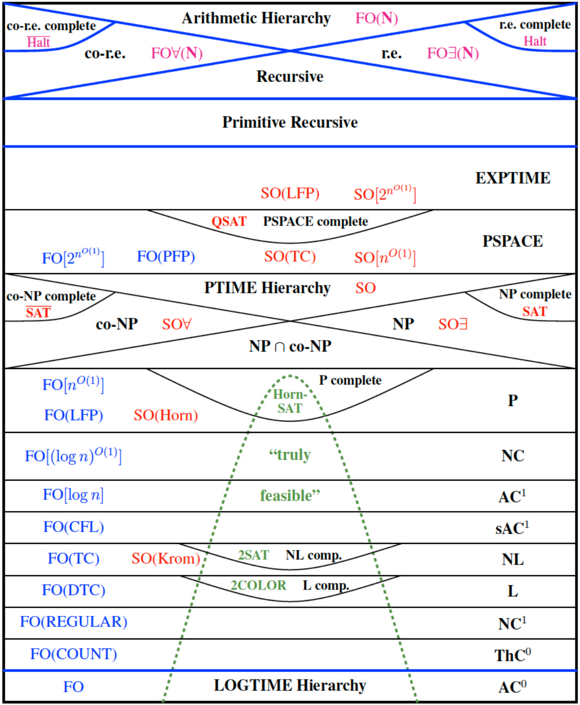

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P is a good mathematical wrapper for "truly feasible".

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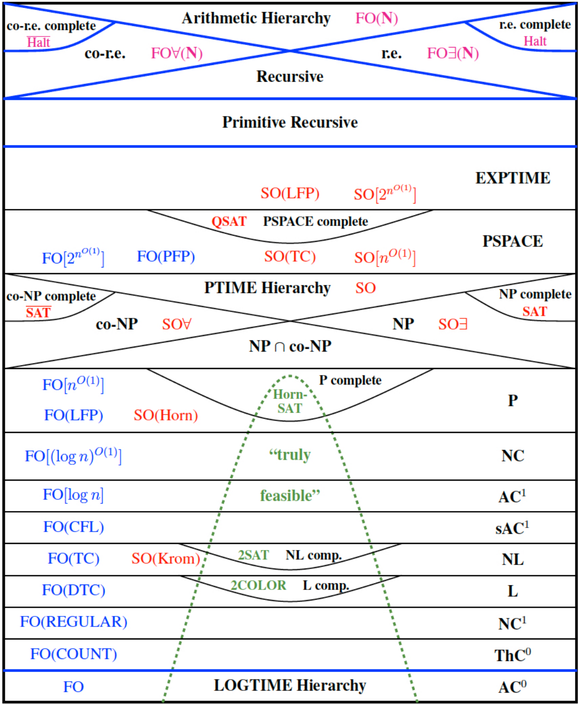


NTIME[$t(n)$]: a mathematical fiction

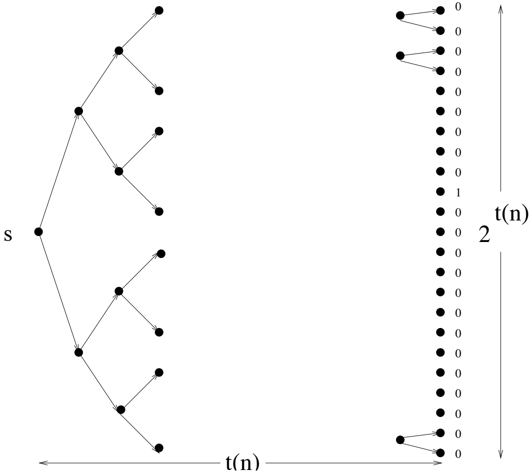
input w, $|w| = n$

N accepts w

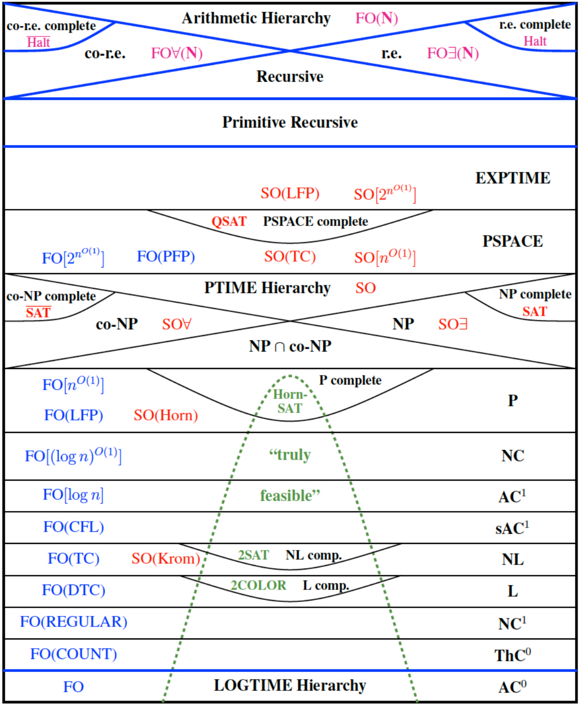
if at least

one of the $2^{t(n)}$

paths accepts.



$$\mathrm{NP} = \bigcup_{k=1}^{\infty} \mathrm{NTIME}[n^k]$$



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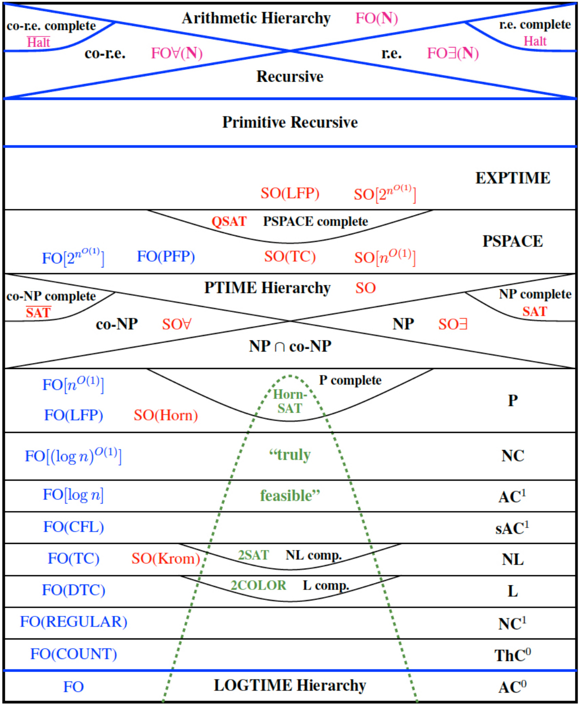
Many optimization problems we want to solve are NP complete.

SAT, TSP, 3-COLOR, CLIQUE, . . .



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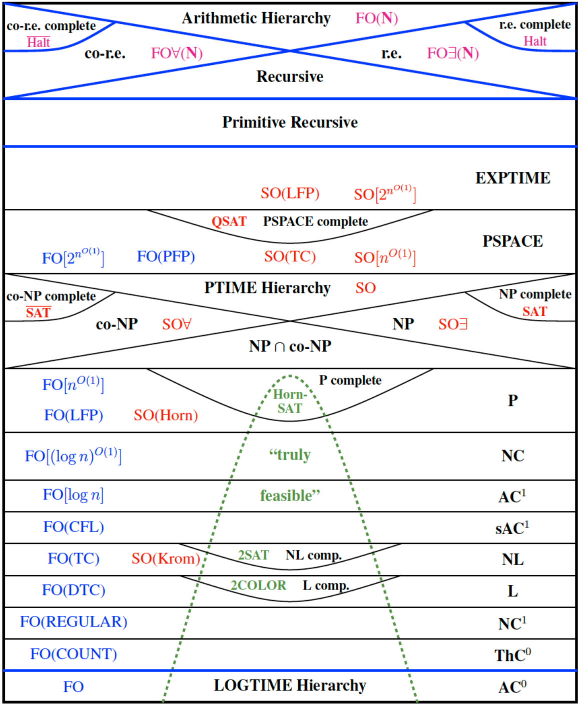
As descision problems, all NP complete problems are isomorphic.



Descriptive Complexity

$$\begin{array}{c}\textbf{Input}\\ q_1\ q_2\ \cdots\ q_n\end{array} \mapsto \boxed{\textbf{Computation}} \mapsto \begin{array}{c}\textbf{Answer}\\ a_1\ a_2\ \cdots\ a_i\ \cdots\ a_m\end{array}$$

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Restrict attention to the complexity of computing individual bits of the output, i.e., **decision problems.**

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There is a constructive isomorphism between these two approaches.

Think of the Input as a Finite Logical Structure

Graph $\qquad G = (\{v_1, \ldots, v_n\}, \leq, E, s, t)$

$\Sigma_g = (E^2, s, t)$

Binary String $\qquad \mathcal{A}_w = (\{p_1, \ldots, p_8\}, \leq, S)$

$S = \{p_2, p_5, p_7, p_8\}$

$\Sigma_s = (S^1) \qquad w = 01001011$

First-Order Logic

input symbols:	from Σ
variables:	$x, y, z, \ldots$
boolean connectives:	$\wedge, \vee, \neg$
quantifiers:	$\forall, \exists$
numeric symbols:	$=, \leq, +, \times, min, max$

$$\alpha \equiv \forall x \exists y (E(x,y)) \in \mathcal{L}(\Sigma_g)$$

$$\beta \equiv \exists x \forall y (x \leq y \wedge S(x)) \in \mathcal{L}(\Sigma_s)$$

$$\beta \equiv S(min) \in \mathcal{L}(\Sigma_s)$$

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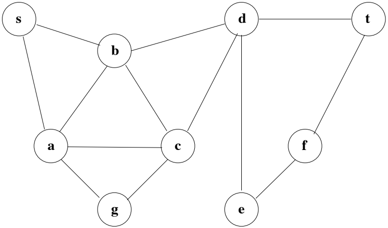
In this setting, with the structure of interest being the **finite input**, FO is a weak, low-level complexity class.

Second-Order Logic: FO plus Relation Variables

$$\Phi_{3\text{color}} \equiv \exists R^1 G^1 B^1 \forall x\, y\, ((R(x) \vee G(x) \vee B(x)) \wedge (E(x,y) \rightarrow$$
$$(\neg(R(x) \wedge R(y)) \wedge \neg(G(x) \wedge G(y)) \wedge \neg(B(x) \wedge B(y)))))$$



Second-Order Logic: FO plus Relation Variables

Fagin's Theorem: NP $=$ SO$\exists$

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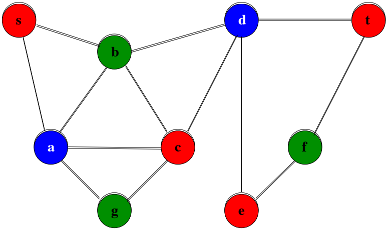

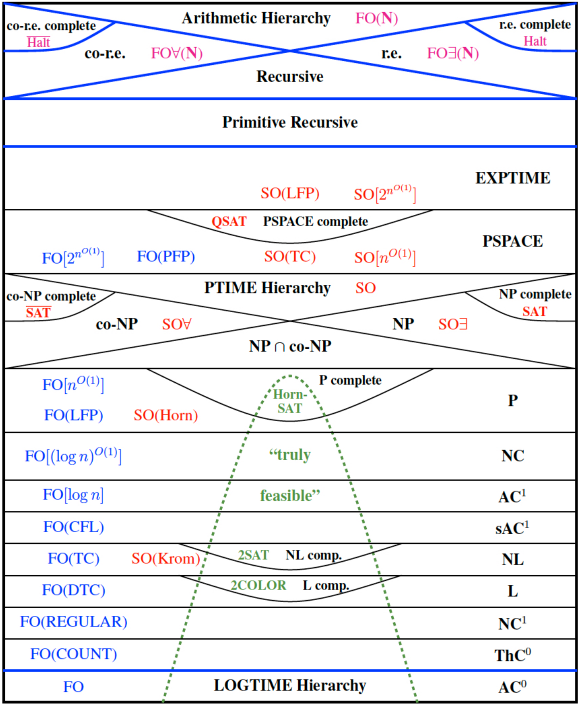

Arithmetic Hierarchy
FO(N)
co-r.e. complete
$\overline{\text{Halt}}$
r.e. complete
Halt
co-r.e.
FO∀(N)
r.e.
FO∃(N)
Recursive
Primitive Recursive
EXPTIME
SO(LFP)
$SO[2^{n^{O(1)}}]$
QSAT
PSPACE complete
PSPACE
$FO[2^{n^{O(1)}}]$
FO(PFP)
SO(TC)
$SO[n^{O(1)}]$
PTIME Hierarchy
SO
co-NP complete
$\overline{\text{SAT}}$
NP complete
SAT
co-NP
SO∀
NP
SO∃
NP ∩ co-NP
$FO[n^{O(1)}]$
P complete
Horn-SAT
P
FO(LFP)
SO(Horn)
$FO[(\log n)^{O(1)}]$
"truly
NC
$FO[\log n]$
feasible"
AC^1
FO(CFL)
sAC^1
FO(TC)
SO(Krom)
2SAT
NL comp.
NL
FO(DTC)
2COLOR
L comp.
L
FO(REGULAR)
NC^1
FO(COUNT)
ThC^0
FO
LOGTIME Hierarchy
AC^0

Addition is First-Order

$$Q_+ : \mathrm{STRUC}[\Sigma_{AB}] \to \mathrm{STRUC}[\Sigma_s]$$

$$\begin{array}{lcccccc}
A & & a_1 & a_2 & \ldots & a_{n-1} & a_n \\
B & + & b_1 & b_2 & \ldots & b_{n-1} & b_n \\
\cline{3-7}
S & & s_1 & s_2 & \ldots & s_{n-1} & s_n
\end{array}$$

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$$\begin{aligned}
C(i) \quad \equiv \quad & (\exists j > i)\Big(A(j) \wedge B(j) \ \wedge \\
& \qquad (\forall k.j > k > i)(A(k) \vee B(k))\Big)
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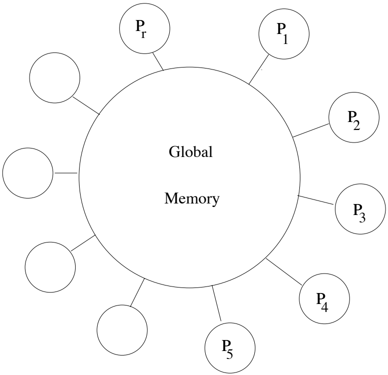
$$\begin{aligned} C(i) \quad &\equiv \quad (\exists j > i)\Big(A(j) \land B(j) \;\land \\ &\qquad (\forall k.j > k > i)(A(k) \lor B(k))\Big) \end{aligned}$$

$$Q_+(i) \quad \equiv \quad A(i) \oplus B(i) \oplus C(i)$$

Parallel Machines:

$$\mathrm{CRAM}[t(n)] \; = \mathrm{CRCW\text{-}PRAM\text{-}TIME}[t(n)]\text{-}\mathrm{HARD}[n^{O(1)}]$$



Parallel Machines: Quantifiers are Parallel

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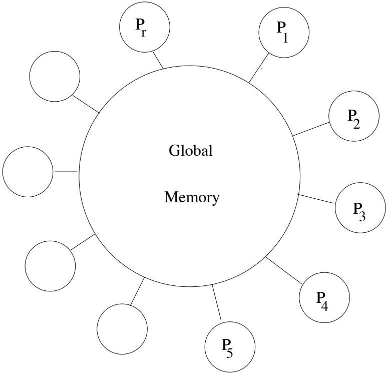
Assume array $A[x] : x = 1, \ldots, r$ in memory.



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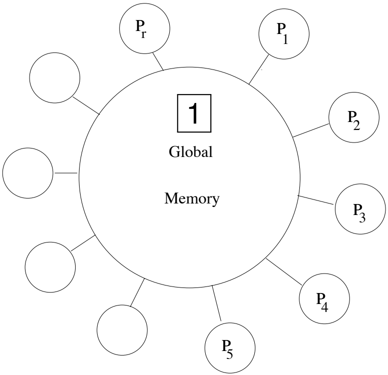
$\forall x(A(x)) \equiv$ **write**(1);



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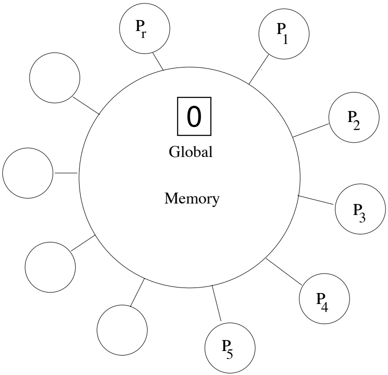


FO

=

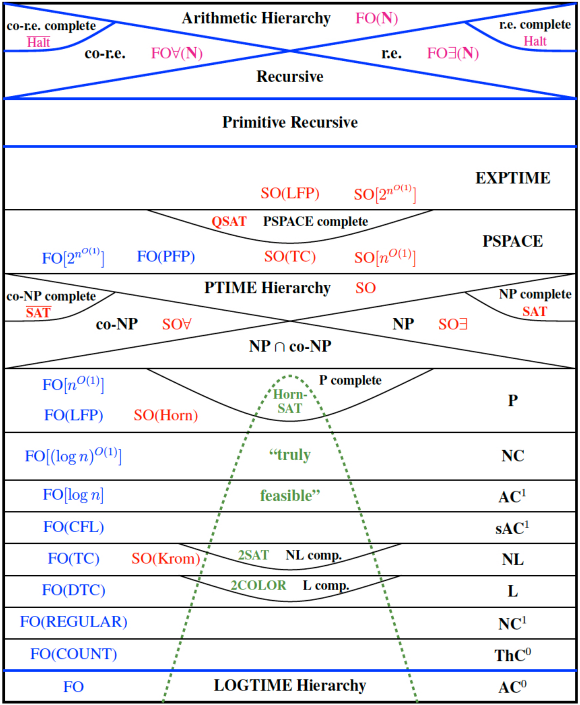
CRAM[1]

=

AC^0

=

Logarithmic-Time Hierarchy



Inductive Definitions and Least Fixed Point

$$\text{REACH} \;=\; \left\{G, s, t \;\middle|\; s \xrightarrow{\star} t\right\}$$

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$\text{REACH} = \{G, s, t \mid s \xrightarrow{\star} t\}$ $\qquad$ $\text{REACH} \notin \mathrm{FO}$

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$$E^\star(x,y) \quad \stackrel{\text{def}}{=} \quad x = y \ \lor\ E(x,y) \ \lor\ \exists z(E^\star(x,z) \land E^\star(z,y))$$

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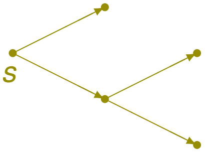

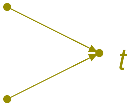

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$$E^\star(x,y) \stackrel{\text{def}}{=} x = y \lor E(x,y) \lor \exists z(E^\star(x,z) \land E^\star(z,y))$$

$$\varphi_{tc}(R,x,y) \equiv x = y \lor E(x,y) \lor \exists z(R(x,z) \land R(z,y))$$

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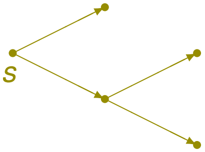

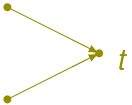

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$$E^\star = (\mathrm{LFP}\varphi_{tc})$$

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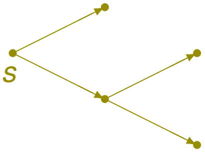

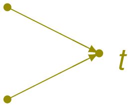

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$$\varphi_{tc}^G : \text{binRel}(G) \quad \to \quad \text{binRel}(G) \qquad \text{is a monotone operator}$$

$$G \in \text{REACH} \ \Leftrightarrow \ G \models (\mathrm{LFP}\varphi_{tc})(s,t) \qquad E^\star \ = \ (\mathrm{LFP}\varphi_{tc})$$

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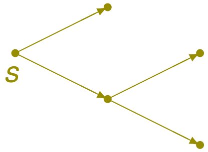

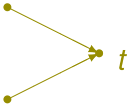

Tarski-Knaster Theorem

Thm. If $\varphi : \mathrm{Rel}^k(G) \to \mathrm{Rel}^k(G)$ is monotone, then $\mathrm{LFP}(\varphi)$ exists and can be computed in P.

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Suppose $\varphi(F) = F$. $\quad$ By induction on r, for all r, $I^r \subseteq F$.

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base case: $\quad I^0 = \emptyset \subseteq F$.

Tarski-Knaster Theorem

Thm. If $\varphi : \mathrm{Rel}^k(G) \to \mathrm{Rel}^k(G)$ is monotone, then $\mathrm{LFP}(\varphi)$ exists and can be computed in P.

proof: Monotone means, for all $R \subseteq S$, $\quad \varphi(R) \subseteq \varphi(S)$.

Let $I^0 \stackrel{\text{def}}{=} \emptyset$; $\quad I^{r+1} \stackrel{\text{def}}{=} \varphi(I^r)$ $\quad$ Thus, $\emptyset = I^0 \subseteq I^1 \subseteq \cdots \subseteq I^t$.

Let t be min such that $I^t = I^{t+1}$. Note that $t \leq n^k$ where $n = |V^G|$. $\quad \varphi(I^t) = I^t$, $\quad$ so I^t is a fixed point of φ.

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Thus $I^t \subseteq F$ $\quad$ and $\quad I^t = \mathrm{LFP}(\varphi)$. $\square$

Inductive Definition of Transitive Closure

$$\varphi_{tc}(R, x, y) \quad \equiv \quad x = y \ \vee \ E(x, y) \ \vee \ \exists z(R(x, z) \wedge R(z, y))$$

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Next we will show that $\mathrm{IND}[t(n)] \ = \ \mathrm{FO}[t(n)]$.

$\varphi_{tc}(R, x, y) \equiv x = y \vee E(x, y) \vee \exists z\,(R(x, z) \wedge R(z, y))$

1. Dummy universal quantification for base case:

$$
\begin{aligned}
\varphi_{tc}(R, x, y) &\equiv (\forall z.M_1)(\exists z)(R(x, z) \wedge R(z, y)) \\
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2. Using $\forall$, replace two occurrences of R with one:

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\varphi_{tc}(R, x, y) &\equiv (\forall z.M_1)(\exists z)(\forall uv.M_2)R(u, v) \\
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3. Requantify x and y.

$$M_3 \equiv (x = u \wedge y = v)$$

$$\varphi_{tc}(R,x,y) \equiv [\,(\forall z.M_1)(\exists z)(\forall uv.M_2)(\exists xy.M_3)\,]\,R(x,y)$$

Every FO inductive definition is equivalent to a quantifier block.

$\mathrm{QB}_{tc} \equiv [(\forall z.M_1)(\exists z)(\forall uv.M_2)(\forall xy.M_3)]$

$$\varphi_{tc}(R, x, y) \equiv [(\forall z.M_1)(\exists z)(\forall uv.M_2)(\exists xy.M_3)]R(x, y)$$

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Thus, for any structure $\mathcal{A} \in \mathrm{STRUC}[\Sigma_g]$,

$$\begin{aligned} \mathcal{A} \in \mathrm{REACH} \quad &\Leftrightarrow \quad \mathcal{A} \models (\mathrm{LFP}\varphi_{tc})(s,t) \\ &\Leftrightarrow \quad \mathcal{A} \models ([\mathrm{QB}_{tc}]^{\lceil 1+\log \|\mathcal{A}\| \rceil}\,\mathbf{false})(s,t) \end{aligned}$$

$$
\begin{aligned}
\mathrm{CRAM}[t(n)] &= \text{concurrent parallel random access machine;}\\
&\ \text{polynomial hardware, parallel time } O(t(n))\\[2ex]
\mathrm{IND}[t(n)] &= \text{first-order, depth } t(n) \text{ inductive definitions}\\[2ex]
\mathrm{FO}[t(n)] &= t(n) \text{ repetitions of a block of restricted quantifiers:}\\[2ex]
\mathrm{QB} &= [(Q_1 x_1 . M_1) \cdots (Q_k x_k . M_k)]; \quad M_i \text{ quantifier-free}\\[2ex]
\varphi_n &= \underbrace{[\mathrm{QB}][\mathrm{QB}] \cdots [\mathrm{QB}]}_{t(n)} M_0
\end{aligned}
$$

parallel time = inductive depth = QB iteration

Thm. For all constructible, polynomially bounded $t(n)$,

$$\mathrm{CRAM}[t(n)] \;=\; \mathrm{IND}[t(n)] \;=\; \mathrm{FO}[t(n)]$$

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Thm. For all $t(n)$, even beyond polynomial,

$$\mathrm{CRAM}[t(n)] \quad = \quad \mathrm{FO}[t(n)]$$

For $t(n)$ poly bdd,

$\mathrm{CRAM}[t(n)]$

$=$

$\mathrm{IND}[t(n)]$

$=$

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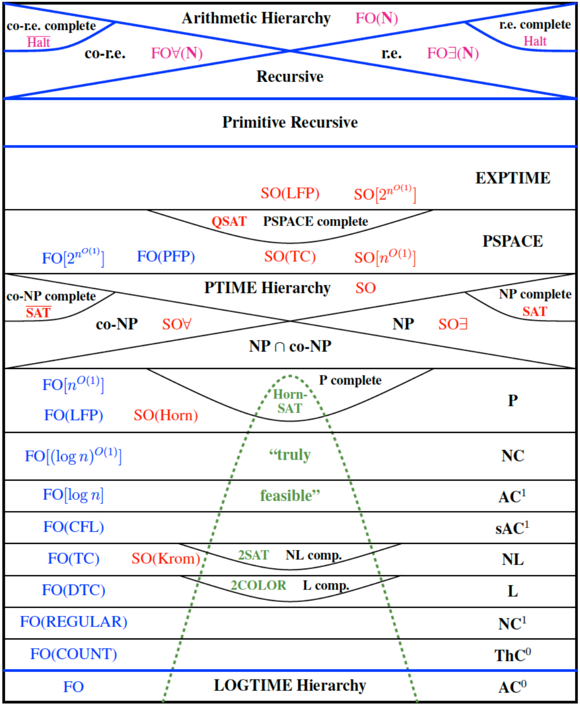


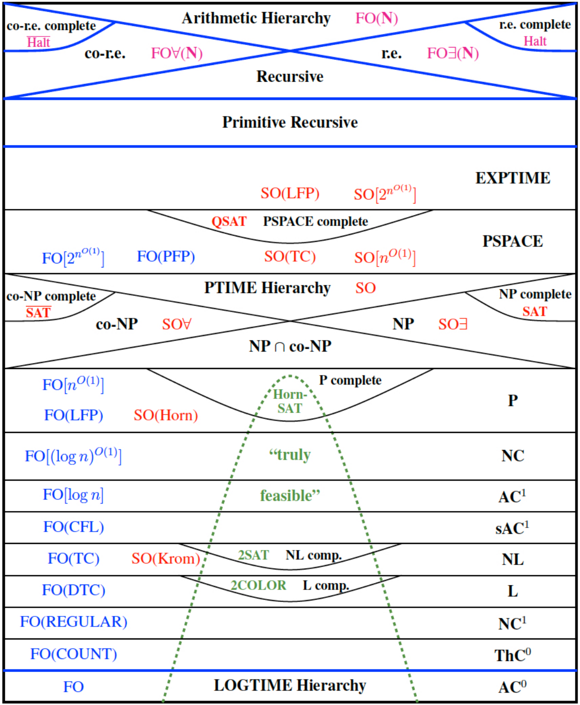
Remember that

for all $t(n)$,

$$\mathrm{CRAM}[t(n)]$$

$$=$$

$$\mathrm{FO}[t(n)]$$



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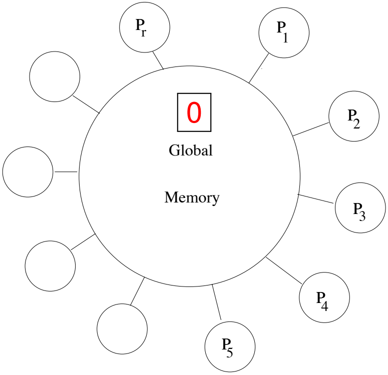
A second-order variable of arity r is n^r bits, corresponding to 2^{n^r} gates.

SO: Parallel Machines with Exponential Hardware

Given φ with n variables and m clauses, is $\varphi \in$ 3-SAT?



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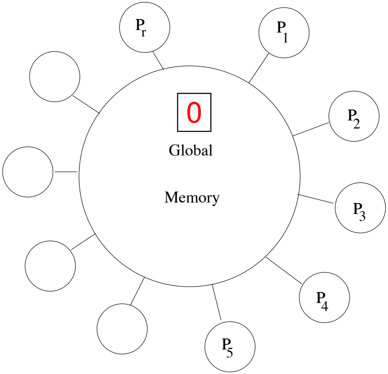

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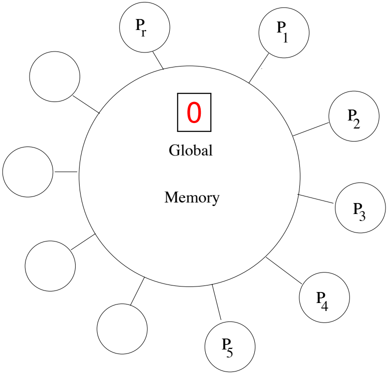

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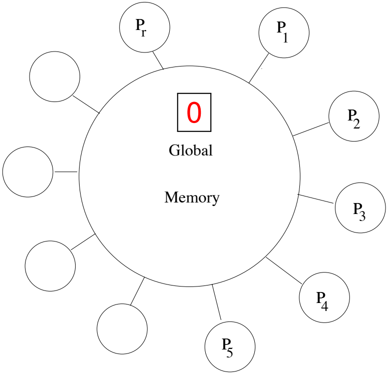

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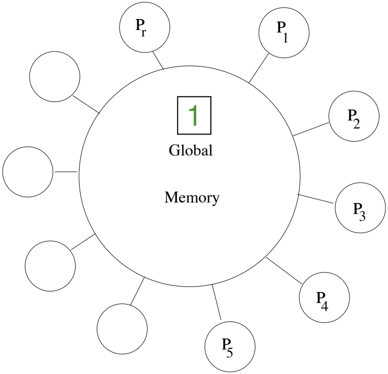

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Thm. $\mathrm{SO}[t(n)] = \mathrm{CRAM}[t(n)]\text{-}\mathrm{HARD}[2^{n^{O(1)}}]$.

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proof: $\mathrm{SO}[t(n)]$ is like $\mathrm{FO}[t(n)]$ but using a quantifier block containing both first-order and second-order quantifiers.
The proof is similar to $\mathrm{FO}[t(n)] = \mathrm{CRAM}[t(n)]$. $\square$

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\begin{array}{ccccc}
\mathrm{SO} & = & \text{PTIME Hierarchy} & = & \mathrm{CRAM}[1]\text{-}\mathrm{HARD}[2^{n^{O(1)}}] \\
\mathrm{SO}[n^{O(1)}] & = & \mathrm{PSPACE} & = & \mathrm{CRAM}[n^{O(1)}]\text{-}\mathrm{HARD}[2^{n^{O(1)}}] \\
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$=$

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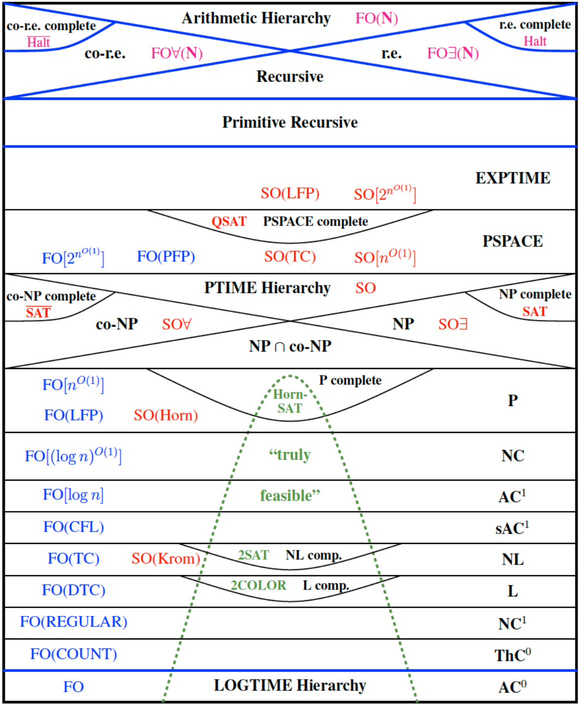

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- **Dichotomy:** "Natural" problems are complete for important compexity classes [FV99, S78, ABISV09].

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Lemma: Let $f : A \leq_{\mathrm{p}} B$ and $g : B \leq_{\mathrm{p}} A$ where f and g are 1:1 length-increasing functions. Assume also that f and g have left inverses in FP. Then A is p-isomorphic to B.

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Proof: Since f, g are length-increasing, the ancestor chains are linear in length. Thus, the isomorphism, h, can be defined as in the SB Thm, but now it can be computed in ptime. $\square$

Def. $A \subseteq \Sigma^*$ has **p-time padding functions** if $\exists e, d \in \mathrm{FP}$ s.t.

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Thus, all the NP complete sets in [GJ] are p-isomorphic. $\square$

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The rest of the proof is similar to proof from [BH77]. $\square$

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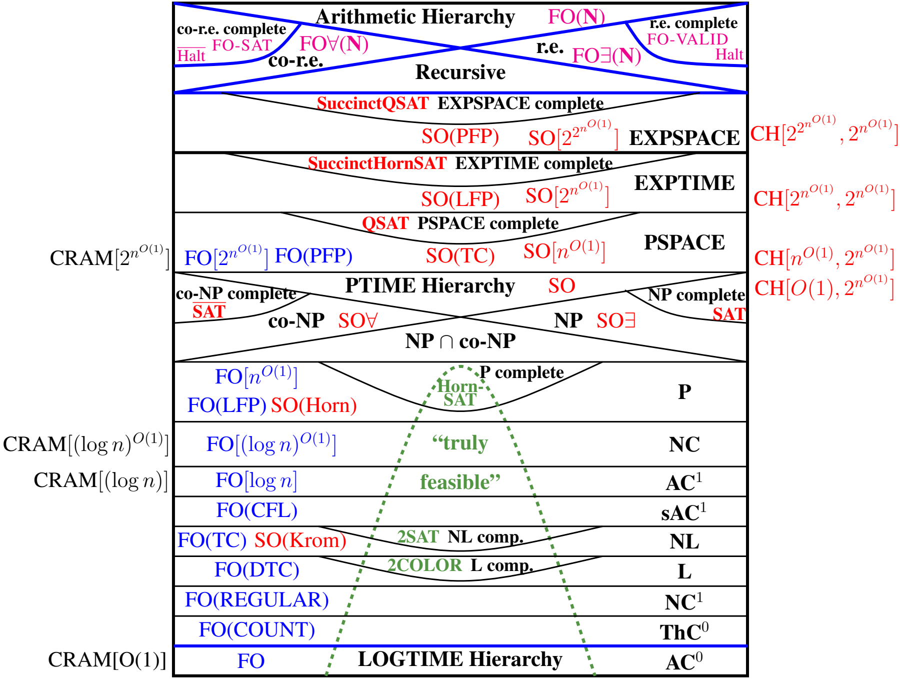
- **Logical** and **Algebraic** reasons, e.g., CSP.


Arithmetic Hierarchy
FO(N)
co-r.e. complete
r.e. complete
FO-SAT
FO∀(N)
FO-VALID
Halt
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PTIME Hierarchy
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CH[$O(1), 2^{n^{O(1)}}$]
co-NP complete
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CRAM[$(\log n)^{O(1)}$]
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“truly
NC
CRAM[$(\log n)$]
FO[$\log n$]
feasible”
AC^1
FO(CFL)
sAC^1
FO(TC) SO(Krom)
2SAT NL comp.
NL
FO(DTC)
2COLOR L comp.
L
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First-Order Isomorphism Theorem [Agrawal01] For nice complexity classes, all complete sets via fops are first-order isomorphic.