
CMPSCI 240: Reasoning Under Uncertainty

Final Exam

May 4, 2015.

Name: _____ ID: _____

Instructions:

- Answer the questions directly on the exam pages.
- Show all your work for each question. Giving more detail including comments and explanations can help with assignment of partial credit.
- If the answer to a question is a number, you may give your answer using arithmetic operations, such as addition, multiplication, and factorial (e.g., “ $9 \times 35! + 2$ ” or “ $0.5 \times 0.3 / (0.2 \times 0.5 + 0.9 \times 0.1)$ ” is fine) unless the problem says otherwise.
- If you need extra space, use the back of a page.
- No books, notes, calculators or other electronic devices are allowed. Any cheating will result in a grade of 0.
- If you have questions during the exam, raise your hand.
- The formulas for some standard random variables can be found on the last page.

Question	Value	Points Earned
1	10	
2	10	
3	10	
4	10	
5	8+2	
6	8+2	
7	8+2	
Total	64+6	

Question 1. (10 points) Indicate whether each of the following five statements is TRUE or FALSE. No justification is required.

1.1 (2 points): *If A and B are disjoint events then $P(A \cup B) = P(A) + P(B)$.*

1.2 (2 points): *Adding a parity bit when transmitting information can help increase the probability that transmission errors will be detected.*

1.3 (2 points): *For any random variable X , $P(X \leq 10) \leq P(X \leq 20)$.*

1.4 (2 points): *Huffman encoding will always encode a message that has probability greater than 0.5 by a binary string of length one.*

1.5 (2 points): *If X is a binomial distribution with parameters N and p then $P(X \geq k) \leq \binom{N}{k} p^k$.*

Question 2. (10 points) Let X be a random variable that evaluates to -1 , 0 , or 1 where

$$P(X = -1) = 1/4 \quad P(X = 0) = 1/2 \quad P(X = 1) = 1/4 .$$

2.1 (3 points): Compute the following and fully simplify your answers:

$$E(X) = \qquad E(X^2) = \qquad \text{var}(X) =$$

2.2 (3 points): Let $Y = 10X + 5$. Compute the following and fully simplify your answers:

$$P(Y = 15) = \qquad E(Y) = \qquad \text{var}(Y) =$$

2.3 (2 points): Compute the following and fully simplify your answers:

$$E(X^3) = \qquad \text{cov}(X, X^2) =$$

Are X and X^2 independent?

2.4 (2 points): Let Z be another random variable that evaluates to -1 , 0 , or 1 . What is the probability mass function that makes $\text{var}(Z)$ as large as possible:

$$P(Z = -1) = \qquad P(Z = 0) = \qquad P(Z = 1) =$$

Question 3. (10 points) Alice and Bob are playing Rock-Paper-Scissors. In this game, each player simultaneously forms a shape with their hand to indicate either a *rock*, a piece of *paper*, or a pair of *scissors*. If Alice and Bob pick rock and paper then the one who plays paper gets \$1 from the other player. If they pick paper and scissors then the one who plays scissors gets \$1 from the other player. If they pick scissors and rock then the one who plays rock gets \$1 from the other player. If both pick the same option they both win \$0.

3.1 (2 points): *What's the pay-off matrix for this game? Hint: Rows should correspond to Alice's options and the entries should indicate Alice's profit.*

3.2 (2 points): *If Bob always picks "rock", what should Alice do to maximize her winnings?*

3.3 (2 points): *Suppose Bob picks "rock" with probability $1/4$, "paper" with probability $3/8$, and "scissors" with probability $3/8$. If Alice knows this, what should she do to maximize her winnings? What is her expected winnings if she does this.*

Alice's Expected Winnings if She Plays Rock :

Alice's Expected Winnings if She Plays Paper :

Alice's Expected Winnings if She Plays Scissors :

Alice's Best Strategy :

3.4 (2 points): *Find a Nash equilibrium for this game. No justification required.*

Alice's Strategy :

Bob's Strategy :

3.5 (2 points): *Indicate whether the following statement is TRUE or FALSE: Every game where each player has a finite number of options has at least one Nash equilibrium.*

Question 4. (10 points) It's a holiday and you can't remember if the buses are running. You have two hypotheses:

$$H_1 = \text{"the buses are running."}$$

$$H_2 = \text{"the buses are not running."}$$

The priors for these hypotheses are $P(H_1) = 3/5$ and $P(H_2) = 2/5$. Let T be the number of hours you need to wait for a bus. If the buses are running today, T is a geometric random variable with parameter $1/2$ and so you'll wait exactly one hour with probability $1/2$, exactly two hours with probability $1/4$ etc. If the buses aren't running today, then T is always 24.

4.1 (4 points): *What are the values:*

$$P(T = 3|H_1) = \qquad P(T = 3|H_2) =$$

$$P(T \geq 2|H_1) = \qquad P(T \geq 2|H_2) =$$

4.2 (2 points): *If event $\{T = 3\}$ is observed, which is the MAP hypothesis? Show your working.*

4.3 (2 points): *If event $\{T \geq 2\}$ is observed, which is the MAP hypothesis? Show your working.*

4.4 (2 points): *How long do you expect to wait for a bus? Show your working.*

Question 5. (10 points) Suppose that a professor sets a quiz with four TRUE/FALSE questions and suppose that a student just randomly guesses each answer, i.e., for each question she writes “TRUE” with probability $1/2$ and she writes “FALSE” with probability $1/2$.

5.1 (4 points): Let X be the number of questions she gets correct. For example, $P(X = 0) = 1/2^4$ since $X = 0$ corresponds to getting none of the answers correct. Compute the following:

$$P(X = 4) =$$

$$P(X = 3) =$$

$$P(X \geq 3) =$$

$$E(X) =$$

5.2 (4 points): Suppose the 1st question is worth 1 point, the 2nd question is worth 2 points, the 3rd question is worth 4 points and the 4th question is worth 8 points. Let Y be the total number of points she gets. Compute the following:

$$P(Y = 0) =$$

$$P(Y = 7) =$$

$$P(Y = 1|X = 1) =$$

$$E(Y) =$$

5.3 (2 points): **Extra Credit:** Suppose the professor gives the hint that an even number of the correct answers are TRUE. The student adopts a new strategy: she guesses the first three answers and then chooses her last answer such that an even number of her answers are TRUE. Let Z be the number of questions she gets correct and let A be the event her answer to the last question is correct. Compute the following:

$$P(A) =$$

$$E(Z) =$$

Question 6. (10 points) Suppose a class of 90 students has three discussion sections and each student is assigned to a random discussion section, i.e., each student is equally likely to be in any of the three discussion sections and the discussion sections for the students are chosen independently.

6.1 (2 points): Suppose Akshay, Beth, and Chao are three students in the class. What's the probability that they are all in the same discussion section?

6.2 (2 points): What's the probability Akshay, Beth, and Chao are in different discussion sections?

6.3 (2 points): Let D be the number of students in the first discussion section. Compute:

$$E(D) = \qquad \qquad \qquad \text{var}(D) =$$

6.4 (2 points): By applying the Chebyshev bound, show an upper bound on the value of $P(D \geq 70)$.

6.5 (2 points): **Extra Credit.** Using your answer for the above question, show a lower bound on the probability that every discussion section has at most 70 students.

Question 7. (10 points) Assume that a cat is in one of two states, s_1 = “Sleeping” or s_2 = “Awake”. After an hour, a sleeping cat wakes with probability $1/3$ and stays asleep with probability $2/3$. After an hour, an awake cat sleeps with probability $2/3$ and stays awake with probability $1/3$. The transition matrix of the corresponding Markov chain is:

$$M = \begin{pmatrix} 2/3 & 1/3 \\ 2/3 & 1/3 \end{pmatrix}$$

where the first row and first column corresponds to state s_1 . Let X_t be the random variable corresponding to the state after t steps. The initial state is s_1 , i.e., $X_0 = s_1$.

7.1 (2 points): Draw the transition diagram of the Markov chain and label each edge with the appropriate transition probability.

7.2 (2 points): What are the values of the following probabilities. Fully simplify your answers.

$$P(X_1 = s_1) =$$

$$P(X_2 = s_1) =$$

7.3 (2 points): What’s the steady state distribution of the Markov chain? Justify your answer.

7.4 (2 points): Suppose you observe the event $\{X_4 = s_1\}$ and have two hypotheses: $H_1 = \{X_3 = s_1\}$ and $H_2 = \{X_3 = s_2\}$. Which is the MAP hypothesis? Show your working.

7.5 (2 points): **Extra Credit:** How many steps do you expect to wait until the cat awakes twice (i.e., she awakes, then falls asleep again, and then awakes for a second time)? Show your working.

Standard Random Variables

- **Bernoulli Random Variable with parameter $p \in [0, 1]$:**

$$P(X = k) = \begin{cases} 1 - p & \text{if } k = 0 \\ p & \text{if } k = 1 \end{cases}, \quad E(X) = p, \quad \text{var}(X) = p(1 - p)$$

- **Binomial Random Variable with parameters $p \in [0, 1]$ and $N \in \{1, 2, 3, \dots\}$:**

$$\text{For } k \in \{0, 1, 2, \dots, N\}: P(X = k) = \binom{N}{k} p^k (1-p)^{N-k}, \quad E(X) = Np, \quad \text{var}(X) = Np(1-p)$$

- **Geometric Random Variable with parameter $p \in [0, 1]$:**

$$\text{For } k \in \{1, 2, 3, \dots\}: P(X = k) = (1 - p)^{k-1} \cdot p, \quad E(X) = \frac{1}{p}, \quad \text{var}(X) = (1 - p)/p^2$$

- **Poisson Random Variable with parameter $\lambda > 0$:**

$$\text{For } k \in \{0, 1, 2, \dots\}: P(X = k) = e^{-\lambda} \frac{\lambda^k}{k!}, \quad E(X) = \lambda, \quad \text{var}(X) = \lambda$$

- **Discrete Uniform Random Variable with parameters $a, b \in \mathbb{Z}$ and $a < b$:**

$$\text{For } k \in \{a, a + 1, \dots, b\}: P(X = k) = \frac{1}{b - a + 1}, \quad E(X) = \frac{a + b}{2}, \quad \text{var}(X) = \frac{(b - a + 1)^2 - 1}{12}$$

Bayes Formula

- If A_1, \dots, A_n partition Ω then for any event B :

$$P(A_i|B) = \frac{P(A_i \cap B)}{P(B)} = \frac{P(B|A_i)P(A_i)}{\sum_{j=1}^n P(B|A_j)P(A_j)}$$

Covariance

- The covariance of random variables X and Y is $\text{cov}(X, Y) = E(XY) - E(X)E(Y)$.