CMPSCI 240: Reasoning Under Uncertainty
Third Midterm Exam

April 15, 2015.

Name: ______________________________ ID: ____________________

Instructions:

- Answer the questions directly on the exam pages.
- Show all your work for each question. Giving more detail including comments and explanations can help with assignment of partial credit.
- If the answer to a question is a number, you may give your answer using arithmetic operations, such as addition, multiplication, and factorial (e.g., "$9 \times 35! + 2$" or "$0.5 \times 0.3/(0.2 \times 0.5 + 0.9 \times 0.1)$" is fine).
- If you need extra space, use the back of a page.
- No books, notes, calculators or other electronic devices are allowed. Any cheating will result in a grade of 0.
- If you have questions during the exam, raise your hand.
- The formulas for some standard random variables can be found on the last page.

Question	Value	Points Earned
1	10	
2	8+2	
3	10	
4	10	
5	10	
6	10	
Total	58+2	

Question 1. (*10 points*) Indicate whether each of the following five statements is TRUE or FALSE. No justification is required.

1.1 (*2 points*): *If X and Y are independent random variables then $var(X+Y) = var(X)+var(Y)$.*

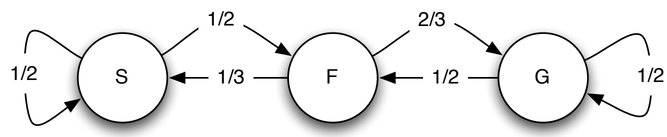
1.2 (*2 points*): *If events A and B satisfy $P(A|B) = P(A)$ then they also satisfy $P(B|A) = P(B)$.*

1.3 (*2 points*): *For any two events A and B, $P(A \cap B) \leq P(A) \times P(B)$.*

1.4 (*2 points*): *Expected number of throws of a 6-sided dice before we've seen all sides is $6 + 1/6$.*

1.5 (*2 points*): *For any two events A and B, $P(A|B^c) = 1 - P(A|B)$.*

Question 2. (*10 points*) The economic cycle can be modeled with a three-state Markov chain as shown below. The three states are shrinking (S), flat (F), and growing (G). Suppose the time steps in the Markov chain correspond to years. The transition diagram is as follows.



2.1 (*2 points*): *Suppose the economy is currently shrinking. What is the probability the economy will be shrinking next year and will become flat the year after that?*

2.2 (*2 points*): *Suppose the economy is currently shrinking. What is the probability the economy will be growing in three years time?*

2.3 (*2 points*): *Circle the two words that apply to the above Markov chain:*

aperiodic periodic reducible irreducible

2.4 (*2 points*): *What is the matrix of transition probabilities corresponding to this Markov chain?*

2.5 (*2 points*): **Extra Credit.** *What are the probabilities in the steady state of the Markov chain?*

Question 3. (*10 points*) You're about to try a new type of ice-cream and have two hypotheses:

$$H_1 = \text{ you will like the ice-cream} \qquad \text{and} \qquad H_2 = \text{ you won't like the ice-cream}$$

Suppose that you've tried 40 different types of ice-cream in your life and so far you've found 20 types that you liked and 20 that you didn't like. You also remember that:

1 of the types you liked included asparagus
2 of the types you liked included banana
0 of the types you didn't like included asparagus
20 of the types you didn't like included banana

Let A and B be the events the new type of ice-cream includes asparagus and banana respectively. You may assume that $P(A \cap B|H_1) = P(A|H_1)P(B|H_1)$ and $P(A \cap B|H_2) = P(A|H_2)P(B|H_2)$.

3.1 (*1 points*): *What values should you use for the priors:*

$$P(H_1) = \qquad\qquad\qquad P(H_2) =$$

3.2 (*2 points*): *What values should you use for the likelihoods:*

$$P(A|H_1) = \qquad\qquad P(B|H_1) = \qquad\qquad P(A|H_2) = \qquad\qquad P(B|H_2) =$$

3.3 (*2 points*): *Which is the maximum a posteri (MAP) hypothesis if the new type of ice-cream includes both asparagus and bananas. Show your working.*

3.4 (*2 points*): *Given that A and B conditionally independent on H_1 is it necessarily true that A^c and B are conditionally independent on H_1? Justify your answer.*

3.5 (*2 points*): *Which is the maximum a posteri (MAP) hypothesis if the new type of ice-cream includes bananas but doesn't include asparagus. Show your working.*

3.6 (*1 points*): *If you wanted to improve your ability to predict the types of ice-cream you like, what additional information would be useful. (A variety of possible suggestions will be accepted.)*

Question 4. (*10 points*) Suppose that five friends, Albert, Bernard, Cheryl, Duane, and Erik, are discussing their birthdays. For this question you should assume that each of their birthdays are equally likely to be any of the 365 days in the year. You should ignore leap years.

4.1 (*2 points*): *What's the probability that the five friends have five different birthdays?*

4.2 (*2 points*): *What's the probability all five friends have the same birthday?*

4.3 (*2 points*): *What's the probability Cheryl's birthday is different from Albert's birthday and Bernard's birthday? Hint: Albert and Bernard may or may not share a birthday with each other.*

In the rest of the question, you should assume that the five friends have five different birthdays. Let H_1 be the event that Albert's birthday is earlier in the year than Bernard's birthday. Let H_2 be the event that Bernard's birthday is earlier in the year than Albert's birthday. Let D be the event that Albert's birthday is earlier in the year than Cheryl's birthday.

4.4 (*2 points*): *What are the values for the following probabilities. Hint: Consider the possible orderings of Albert's, Bernard's and Cheryl's birthdays.*

$P(H_1) =$ $\qquad$ $P(H_2) =$ $\qquad$ $P(D|H_1) =$ $\qquad$ $P(D|H_2) =$

4.5 (*2 points*): *Conditioned on event D, which of H_1 and H_2 is more likely? Show your working.*

Question 5. (*10 points*) This is based on a true story from World War II. Suppose that Germany has numbered their tanks and has printed these numbers of the sides of the tanks. The numbers are $1, 2, 3, \ldots, t$ where t is unknown. The Allied forces want to learn the value of t, i.e., the number of tanks that Germany has built. They have narrowed it down to two hypotheses:

$$H_1 = \text{"there are } t = 100 \text{ tanks"}$$

$$H_2 = \text{"there are } t = 200 \text{ tanks"}$$

and the priors for these hypotheses are $P(H_1) = 1/10$ and $P(H_2) = 9/10$. Each day an Allied spy reports the number of a tank that they have observed. You should assume that the tank is equally likely to be any of the tanks that have been built. Today the spy reports that a tank was sighted with the number 50. Call this event D_1.

5.1 (*2 points*): *What values would you pick for the following likelihoods.*

$$P(D_1|H_1) = \qquad\qquad P(D_1|H_2) =$$

5.2 (*2 points*): *What is the value of the following probability:*

$$P(D_1) =$$

5.3 (*2 points*): *Which is the maximum a posteri (MAP) hypothesis given D_1? Show your working.*

5.4 (*2 points*): *Tomorrow you receive a report of a tank with number 75. Call this event D_2. Which is the MAP hypothesis given D_1 and D_2? Show your working.*

5.5 (*2 points*): *If H_1 is true, how many reports (including the report from today and tomorrow) from the spy will it take to convince you that H_1 is true?*

Question 6. (*10 points*) Every year there is an 80% chance that you are exposed to the flu virus. If you are exposed to the flu virus, there is a 50% chance you'll develop multiple flu symptoms. If you are not exposed there's a 0% chance you'll develop multiple flu symptoms. If you develop multiple flu symptoms, there's a 75% chance that you have a high temperature. Assume there's a 10% chance you'll have a high temperature even if you don't develop multiple flu symptons. Let $V = 1$ if you are exposed to the virus and $V = 0$ otherwise. Let $S = 1$ if you develop multiple flu symptoms and $S = 0$ otherwise. Let $T = 1$ if you have a high temperature and $T = 0$ otherwise. You should assume that V and T are independent conditioned on S.

6.1 (*3 points*): *Enter the values for the following probabilities:*

$$
\begin{aligned}
P(V = 1, S = 0, T = 1) &= \\
P(V = 1, S = 1, T = 1) &= \\
P(V = 1, T = 1) &=
\end{aligned}
$$

6.2 (*3 points*): *Enter the values for the following probabilities:*

$$
\begin{aligned}
P(S = 0) &= \\
P(S = 1) &= \\
P(T = 1) &=
\end{aligned}
$$

6.3 (*2 points*): *Are S and T independent? Justify your answer numerically.*

6.4 (*2 points*): *Draw the Bayesian network for the random variables V, S, and T.*

Standard Random Variables

- **Bernoulli Random Variable with parameter** $p \in [0,1]$:

$$P(X=k) = \begin{cases} 1-p & \text{if } k=0 \\ p & \text{if } k=1 \end{cases} \quad , \quad E(X)=p \quad , \quad var(X)=p(1-p)$$

- **Binomial Random Variable with parameters** $p \in [0,1]$ **and** $N \in \{1,2,3,\ldots\}$:

For $k \in \{0,1,2,\ldots,N\}$: $P(X=k) = \binom{N}{k} p^k \, (1-p)^{N-k}$, $E(X) = Np$, $var(X) = Np(1-p)$

- **Geometric Random Variable with parameter** $p \in [0,1]$:

For $k \in \{1,2,3,\ldots\}$: $P(X=k) = (1-p)^{k-1} \cdot p$, $E(X) = \dfrac{1}{p}$, $var(X) = (1-p)/p^2$

- **Poisson Random Variable with parameter** $\lambda > 0$:

For $k \in \{0,1,2,\ldots\}$: $P(X=k) = e^{-\lambda} \dfrac{\lambda^k}{k!}$, $E(X) = \lambda$, $var(X) = \lambda$

- **Discrete Uniform Random Variable with parameters** $a, b \in \mathbb{Z}$ **and** $a < b$:

For $k \in \{a, a+1, \ldots b\}$: $P(X=k) = \dfrac{1}{b-a+1}$, $E(X) = \dfrac{a+b}{2}$, $var(X) = \dfrac{(b-a+1)^2-1}{12}$

Bayes Formula

- If $A_1, \ldots, A_n$ partition Ω then for any event B:

$$P(A_i|B) = \frac{P(A_i \cap B)}{P(B)} = \frac{P(B|A_i)P(A_i)}{\sum_{j=1}^n P(B|A_j)P(A_j)}$$