
CMPSCI 240: Reasoning Under Uncertainty Final Exam

May 5, 2016.

Name: _____ ID: _____

Instructions:

- Answer the questions directly on the exam pages.
- Show all your work for each question. Providing more detail including comments and explanations can help with assignment of partial credit.
- *Unless the question specifies otherwise*, you may give your answer using arithmetic operations, such as addition, multiplication, “choose” notation and factorials (e.g., “ $9 \times 35! + 2$ ” or “ $0.5 \times 0.3 / (0.2 \times 0.5 + 0.9 \times 0.1)$ ” is fine).
- If you need extra space, use the back of a page.
- No books, notes, calculators or other electronic devices are allowed. Any cheating will result in a grade of 0.
- If you have questions during the exam, raise your hand.
- The formulas for some standard random variables can be found on the last page.

Question	Value	Points Earned
1	10	
2	10	
3	10	
4	10	
5	10	
6	8+2 Extra Credit	
7	8+2 Extra Credit	
Total	66+4 (Extra Credit)	

Question 1. (10 points) Indicate whether each of the following statements is TRUE or FALSE. No justification is required.

1.1 (2 points): For any two random variables X and Y , $\text{var}(X + Y) = \text{var}(X) + \text{var}(Y)$.

1.2 (2 points): For any two events A and B such that $P(B) > 0$ then $P(A|B) \leq 1$.

1.3 (2 points): $\binom{10}{8} < \binom{10}{9}$.

1.4 (2 points): Every game where each player has a finite number of options, has at least one Nash equilibrium.

1.5 (2 points): $P(X > 1) = 1 - p$ if X is a geometric random variable with parameter p .

Question 2. (10 points) Suppose you throw a fair 12-sided dice to get a value from the set

$$\Omega = \{1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12\} .$$

Consider the events

$$A = \{1, 2, 3, 4, 9, 10\} \quad , \quad B = \{3, 4, 5, 6, 9, 10\} \quad , \quad C = \{1, 2, 3, 4, 5, 6, 7, 8\}$$

2.1 (3 points): *What are the following values:*

$$P(A) = \qquad P(B) = \qquad P(A \cap B) =$$

2.2 (1 points): *Are the events A and B independent?*

2.3 (3 points): *What are the following values:*

$$P(A|C) = \qquad P(B|C) = \qquad P(A \cap B|C) =$$

2.4 (1 points): *Are the events A and B independent conditioned on C?*

2.5 (2 points): *Suppose you keep on throwing the dice until you have seen all twelve different values. What is the expected number of times that you will need to throw the dice?*

Question 3. (10 points) A mouse visits 3 locations in your house. It hops from one location to another in the following manner: if one day it is at location $i \in \{1, 2, 3\}$, then on the next day it will remain at that same location with probability $\frac{1}{3}$, and will move to one of the two other locations with equal probability. Let X_n be the random variable corresponding to the mouse's location on day n .

3.1 (2 points): Draw the transition diagram of the Markov chain and label each edge with the appropriate transition probabilities.

3.2 (4 points): Suppose $X_0 = 1$. Compute the following values. For full marks, fully simplify your answers.

$$P(X_1 = 3) =$$

$$P(X_2 = 3) =$$

$$P(X_1 = 3, X_2 = 3) =$$

$$E(X_2) =$$

3.3 (2 points): What is the steady state distribution of the Markov chain? Show working.

3.4 (2 points): Suppose you put a trap at location 2. What is the expected number of days required to trap the mouse?

Question 4. (10 points) Suppose that 1 in 5 emails I receive are spam. When my computer receives spam it puts it in the junk mail folder with probability $5/6$. When my computer receives a message that isn't spam, it puts it in the junk mail folder with probability $1/3$. Define the events:

$S =$ "email is spam" , $J =$ "email gets put in my junk mail folder"

4.1 (2 points): Enter the values for the following probabilities:

$$P(S) = \quad P(S^c) = \quad P(J|S) = \quad P(J|S^c) = \quad P(J^c|S) = \quad P(J^c|S^c) =$$

4.2 (2 points): What's the probability that the next email is spam and is put in the junk mail folder?

4.3 (2 points): What's the probability that the next email is not spam and is put in the junk mail folder?

4.4 (2 points): What's the probability that the next email gets put in the junk mail folder?

4.5 (2 points): What's the probability that an email in the junk mail folder is actually spam?

Question 5. (10 points) Suppose your friend has randomly picked a number from $\{1, 2, 3, 4, 5, 6, 7, 8\}$ and each number is equally likely. You have to guess the number by asking only the following type of question: “Is the number i ?” where $i \in \{1, 2, 3, 4, 5, 6, 7, 8\}$. You friend can only answer yes or no to each question. **Hint:** Without loss of generality, you may assume that you first guess “1”, then “2”, then “3” etc.

5.1 (5 points): Let X denote the number of questions of the above type you need until you guess the number correctly. What is the value of the following quantities? To get full marks your answers must be simplified fully.

$$P(X = 1) =$$

$$P(X = 2) =$$

$$E(X) =$$

$$\text{var}(X) =$$

$$E(X^2) =$$

5.2 (2 points): What bound on $P(|X - 4.5| \geq 3)$ is implied by the Chebyshev bound?

5.3 (2 points): What is the exact value of $P(|X - 4.5| \geq 3)$?

5.4 (1 points): If you are allowed to ask any type of yes/no questions then how many yes/no questions suffice to determine her number?

Question 6. (10 points) In the *Game of Throws*, you may win by throwing a dice. The Braavos Savings Bank is sponsoring a competition. Two players, Arya and Bran, each have a fair four-sided dice where the sides are numbered 1, 2, 3, and 4. Let X be the value when Arya throws her dice and let Y be the value when Bran throws his dice. Let $Z = \min(X, Y)$. If $X < Y$ then Arya receives Z pennies from the bank. If $Y < X$ then Bran receives Z pennies. In other words, the player with the *lowest value* receives this value as a prize. If $X = Y$ then Arya and Bran both get 0 pennies. Arya and Bran never lose money.

6.1 (3 points): What are the values of the following:

$$E(X) =$$

$$P(\text{Arya and Bran both get 0 pennies}) = P(X = Y) =$$

$$P(\text{Arya receives some pennies}) = P(X < Y) =$$

6.2 (3 points): Note that $P(Z = 1) = 7/16$ because there are 16 possible outcomes and in 7 of these the minimum of the two dice rolls is 1. What are the values of the following:

$$P(Z = 2) =$$

$$P(Z = 3) =$$

$$P(Z = 4) =$$

6.3 (2 points): Let A be the number of pennies that Arya wins. What is the value of:

$$E(A) =$$

6.4 (2 points): **Extra Credit.** Suppose Bran decides to use a magic dice that lands “4” with probability $1/2$ and has probability $1/6$ of landing on each of the other three sides. If Arya could fix the side that her dice lands when they throw their dice, which side should she choose in order to maximize the expected number of pennies she wins?

Which side of the dice should Arya choose =

What’s her expected winning in this case =

Question 7. (10 points) You're designing a new digital communication device. When each bit is transmitted, an error occurs (i.e., the bit is flipped) with probability 0.1. Assume that the errors are independent.

7.1 (2 points): *What is the probability that a 6 bit message will contain at most 1 error?*

7.2 (2 points): *Suppose the communication device needs to send three information bits a_1 , a_2 , and a_3 . You add the following parity bit to it.*

$$a_4 = a_1 + a_2 + a_3 \pmod{2}$$

What is the probability that the receiver will either detect if an error has happened or will recover the message correctly? What strategy will the receiver use to detect errors?

7.3 (2 points): *If the receiver has to be able to correct one error, what are the other parity bits you need to add to the message?*

7.4 (2 points): *Suppose you have a set \mathcal{S} of possible binary messages that could be sent. The minimum Hamming distance between any pair of messages from \mathcal{S} is at least 7. What is the maximum number of errors the receiver can be guaranteed to correct?*

7.5 (2 points): **Extra Credit.** *If the bit sent is 0 with probability 1/4 and 1 with probability 3/4, what is the entropy of the bit that is received? To get full marks you should simplify your answer fully. Hint. See formula sheet.*

Standard Random Variables

- **Bernoulli Random Variable with parameter $p \in [0, 1]$:**

$$P(X = k) = \begin{cases} 1 - p & \text{if } k = 0 \\ p & \text{if } k = 1 \end{cases}, \quad E(X) = p, \quad \text{var}(X) = p(1 - p)$$

- **Binomial Random Variable with parameters $p \in [0, 1]$ and $N \in \{1, 2, 3, \dots\}$:**

$$\text{For } k \in \{0, 1, 2, \dots, N\}: P(X = k) = \binom{N}{k} p^k (1-p)^{N-k}, \quad E(X) = Np, \quad \text{var}(X) = Np(1-p)$$

- **Geometric Random Variable with parameter $p \in [0, 1]$:**

$$\text{For } k \in \{1, 2, 3, \dots\}: P(X = k) = (1-p)^{k-1} \cdot p, \quad E(X) = \frac{1}{p}, \quad \text{var}(X) = (1-p)/p^2$$

- **Poisson Random Variable with parameter $\lambda > 0$:**

$$\text{For } k \in \{0, 1, 2, \dots\}: P(X = k) = e^{-\lambda} \frac{\lambda^k}{k!}, \quad E(X) = \lambda, \quad \text{var}(X) = \lambda$$

- **Discrete Uniform Random Variable with parameters $a, b \in \mathbb{Z}$ and $a < b$:**

$$\text{For } k \in \{a, a + 1, \dots, b\}: P(X = k) = \frac{1}{b - a + 1}, \quad E(X) = \frac{a + b}{2}, \quad \text{var}(X) = \frac{(b - a + 1)^2 - 1}{12}$$

Some More Definitions.

- **Entropy:** The entropy of a set of probabilities p_1, p_2, \dots, p_n is defined as

$$H = \sum_{i=1}^n p_i \log_2 \frac{1}{p_i}$$