CMPSCI 240: Homework 4

Due: 8pm Tuesday May. 2nd
For full marks, justify and simplify your answers as much as possible.

Question 1. (*8 points*) Consider a variant of two-finger morra where the winner is still determined by the parity of the sum of the two numbers thrown (if it's even, Alice wins money and if it's odd, Bob wins money), but the amount won or lost is the product of the two numbers. Analyze this game and determine the optimal strategy for each player.

1.1 (*3 points*): *If Alice plays 1 finger with probability p and 2 fingers with probability $1-p$, what's the expected payoff she gets even if Bob knows p and can chose his strategy based on this knowledge? How should Alice choose p such that this payoff is maximized?*

1.2 (*3 points*): *If Bob plays 1 finger with probability q and 2 fingers with probability $1-q$, what's the expected payoff he gets even if Alice knows q and can chose her strategy based on this knowledge? How should Bob choose q such that this payoff is maximized?*

1.3 (*2 points*): *What's the optimal strategy for each player and the expected payoff for each if they play the optimal strategy?*

Question 2. (*8 points*) In this question we consider the *iterated prisoners dilemma*, as discussed in lecture 22. In particular, suppose after each game, we play again with probability p. Suppose you know your opponent will confess on every turn once you have confessed once, but will stay mute until then. In class, we showed that if your strategy was to always confess then your expected payoff is $5 - 5/(1-p)$ whereas if you always stayed mute it would be $-1/(1-p)$. The latter strategy is better is $p > 1/5$.

2.1 (*1 points*): *Let $Y_r = 1$ if there are r or more rounds of the game played. What is the value of $E[Y_r]$ as a function of p and r?* **Hint:** *$E[Y_1] = 1$ and $E[Y_2] = p$.*

2.2 (*3 points*): *What is the expected payoff you receive (as a function of p) if your strategy is to stay mute on the first game and then confess on any subsequent game. Simplify your answer fully.* **Hint:** *The best way to solve this problem (in my opinion; you might find an easier way) is to write your payoff as a linear combination of $Y_1, Y_2, Y_3, \ldots$, and use linearity of expectation but there are other ways to solve it.*

2.3 (*2 points*): *When is this strategy better than the strategy of always confessing?*

2.4 (*2 points*): *When is this strategy better than the strategy of always staying mute?*

Question 3. (*7 points*) Alice and Bob are speeding towards each other on a single-track road: if they both swerve left or both swerve right they will not collide. Otherwise they will collide. They both have car insurance that will cover most but not all of the costs if they collide. However, if they collide, the car that swerves right will have payoff $-\$100$ and the car that swerves left will have payoff $-\$150$. (In general, if you are driving in a country that drives on the right, it makes more sense to swerve right in these circumstances).

3.1 (*2 points*): *Write the pay-off matrix for Alice and Bob. Label the row/columns by "swerve left" and "swerve right" as appropriate.*

3.2 (*2 points*): *What are the pure Nash equilibria for Alice and Bob?*

3.3 (*3 points*): *What is the mixed Nash equilibria for Alice and Bob? Remember to show your working. What is the expected payoffs for each player in this case?*

Question 4.

(*7 points*) Consider a sender is trying to send three information bits a_1, a_2, and a_3 over a noisy channel that has error probability $p = 0.01$. That is, with probability p each bit may be flipped independently from 0 to 1 and vice versa. The sender will also send two or three extra parity bits (defined below) and these define codes of length 5 or 6 respectively. Suppose the receiver incorrectly decodes the message if the number of bits flipped is equal or greater to half the minimum distance of this code.

4.1 (*4 points*): *The sender adds three parity bits to it as follows:*

$$\begin{aligned} a_4 &= a_2 + a_3 \quad (mod\ 2) \\ a_5 &= a_1 + a_3 \quad (mod\ 2) \\ a_6 &= a_1 + a_2 \quad (mod\ 2) \end{aligned}$$

E.g., one of the code words is 001110. *Write out all the codewords. What is the minimum distance of the code? What is the probability that the receiver makes an incorrect decision?*

4.2 (*3 points*): *Suppose the sender only adds two parity bits a_4 and a_5? E.g., one of the code words is* 00111. *Write out all the codewords. What is the minimum distance of the code? What is the probability that the receiver makes an incorrect decision?*

Question 5.

(*8 points*) **Extra Credit.** A professor sets a homework with two yes/no questions. For the first question, the students must work by themselves. For the second question the students split into groups of three; each person in the group tells the others their answer; and each student answers the question with the most popular answer (e.g., if the 3 students answer yes-yes-no then the final answer for each of the 3 students is yes). Assume that the probability that a students knows the correct answer to a question is $2/3$ and there are 90 students in the class.

5.1 (*2 points*): *When students work in a group of three, what is the probability that the majority answer is correct?*

5.2 (*3 points*): *Let X be the number of students who get the correct answer for the first question. What is the mean and variance of X? Using the Chebyshev theorem, bound the value of $P(X \geq 50)$.*

5.3 (*3 points*): *Let Y be the number of students who get the correct answer for the second question. What is the mean and variance of Y? Using the Chebyshev theorem, bound the value of $P(Y \geq 50)$.*