CMPSCI 311: Introduction to Algorithms

Lecture 22: Randomized Algorithms

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Today

Randomized Algorithms + Probability Tools

- Resource Contention
- Minimum Cuts

Announcements

- Updated grades
- Updated schedule
- Programming Assignment

Randomized Algorithms

- So far: deterministic algorithms on worst case inputs.
- Why worst case?
  - Enables precise statements
  - But maybe not reflective of real-world instances.
  - Average-case analysis? What distribution?

Randomized Algorithms

- So far: deterministic algorithms on worst case inputs.
- Why deterministic algorithms?
  - Easier to understand, pretty powerful.
- Two types of randomized algorithms:
  - Fail with some small probability.
  - Always succeed but running time is random.
- How powerful are randomized algorithms?

Resource Contention

tl;dr; Randomization helps with symmetry breaking.

- How do we share a resource in distributed settings?
  - Option #1: Coordination between agents (expensive)
  - Option #2: Randomize access

Resource contention

- n agents $P_1, \ldots, P_n$ and a shared resource.
- At each round, agents can request access to the resource.
  - > 1 requests – conflict, no access.
  - 0 requests – wasted round, no access.
  - 1 request – access granted.
- What is a good decentralized protocol where all agents get regular access?

A centralized solution?

- Round-robin access:
  - Cycle through agents $1, \ldots, n$.
  - No wasted round.
  - Each agent gets 1 access every n rounds.
  - After n rounds all agents have accessed.
- But a lot of coordination!

Decentralized solution

On each round, each agent accesses with probability p.
First question: What choice for p?

- Let $A[i,t]$ be the event that P_i attempts to access on round t

$$\Pr[A[i,t]] = p \qquad \Pr[\overline{A[j,t]}] = (1-p)$$

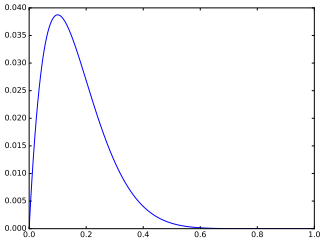
- Let $S[i,t]$ be the event that P_i successfully accesses on round t

$$S[i,t] = A[i,t] \bigcap \left(\bigcap_{j \neq i} \overline{A[j,t]} \right)$$

$$\Pr[S[i,t]] = p(1-p)^{n-1}$$

How to choose p?

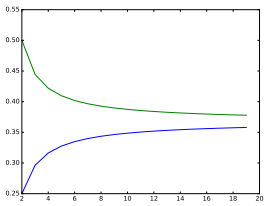
$$f(p) = \Pr[S[i,t]] = p(1-p)^{n-1}$$



$f(p)$ maximized at $p = 1/n$.

Some asymptotics

$$\Pr[S[i,t]] = \frac{1}{n}\left(1 - \frac{1}{n}\right)^{n-1}$$



- $\left(1 - \frac{1}{n}\right)^{n} \to 1/e$ and $\left(1 - \frac{1}{n}\right)^{n-1} \to 1/e$

$$\frac{1}{en} \leq \Pr[S[i,t]] \leq \frac{1}{2n}$$

Decentralized Solution

- Success probability is $\Theta(1/n)$ in one round
- What about for multiple rounds?
  - Def: $F[i,t]$ = event that i unsuccessful through round t.

$$\begin{aligned}\Pr[F[i,t]] &= \Pr[\bigcap_{r=1}^{t} \overline{S[i,r]}] \\ &= \prod_{r=1}^{t} \Pr[\overline{S[i,r]}] \\ &= \left[1 - \frac{1}{n}\left(1 - \frac{1}{n}\right)^{n-1}\right]^t \leq \left(1 - \frac{1}{en}\right)^t\end{aligned}$$

- With $t = \lceil en \rceil$, $\Pr[F[i,t]] \leq \frac{1}{e}$.
- With $t = \lceil en \rceil \cdot c \ln n$, $\Pr[F[i,t]] \leq n^{-c}$.

Comparison

	Centralized	Decentralized
Rounds until individual success	$\Theta(n)$	$\Theta(n \ln n)$ w.h.p.
Rounds until everyone succeeds	$\Theta(n)$	???

Decentralized Solution

- How long until all agents access at least once?
- Let F_t denote event that some agent failed after t rounds

$$\Pr[F_t] = \Pr\left[\bigcup_{i=1}^{n} F[i,t]\right] \leq \sum_{i=1}^{n} \Pr[F[i,t]]$$

Set $t = 2\lceil en \rceil \ln n$

$$\Pr[F_t] \leq \sum_{i=1}^{n} n^{-2} = n^{-1}$$

Theorem

With probability at least $1 - 1/n$ all agents succeed in accessing the resource within $2\lceil en \rceil \ln n$ rounds.

Comparison

	Centralized	Decentralized
Rounds until individual success	$\Theta(n)$	$\Theta(n \ln n)$ w.h.p.
Rounds until everyone succeeds	$\Theta(n)$	$\Theta(n \ln n)$ w.h.p.

Resource Contention Takeaways

- Simple randomization good for symmetry breaking.
- Technical tools:
  - Intersection of independent events
  - Some calculus
  - Union bound

Minimum Cuts

Problem. Given undirected $G = (V, E)$, partition V into sets $A, V \setminus A$ to minimize,

$$\text{cut}(A) = |\{(u,v) \in E, u \in A, v \notin A\}|$$

- Previously, we saw how to compute minimum $s - t$ cut in directed graph.
- How do we compute *global minimum cut*?

Deterministic Algorithm

Idea. Convert into $s - t$ cut in directed graph.

Replace $e = (u, v)$ with directed edges in both directions (with capacity 1).
Pick arbitrary s.
for each other vertex t **do**
    Compute minimum $s - t$ cut.
end for
Return smallest computed $s - t$ cut.

Running Time. n max-flow computations $\Rightarrow O(mn^2)$ at best.

Contraction Algorithm Preliminaries

Def. Multigraph $G = (V, E)$ is a graph that can have parallel edges.
Def. Contracting an edge (u, v) in $G = (V, E)$ produces a new multigraph $G' = (V', E')$

- With new node w instead of u, v ((u, v) edges deleted).
- If (x, u) or $(x, v) \in E$, then $(x, w) \in E'$.
- All other edges preserved.

Contraction Algorithm

$S(v) = \{v\}$ for all $v \in V$.
while $|V| > 2$ **do**
  Pick edge $(u, v) \in E$ uniformly at random.
  Contract edge (u, v) to get G' with new node w
  Set $S(w) \leftarrow S(u) \cup S(v)$.
  Update $G \leftarrow G'$.
end while
Return $S(v)$ for $v \in V$.

Contraction Algorithm Analysis

Theorem. Alg finds global min cut with probability at least $1/\binom{n}{2}$.
Proof. Suppose (A, B) is a global min cut with $\text{cut}(A, B) = k$

- What could go wrong in the first step?
  - Select (u, v) where $u \in A, v \in B$.

$$\Pr[\text{mistake in round 1}] = \Pr[\text{select } u \in A, v \in B] = \frac{k}{\#\text{ of edges}}$$

  - $\#$ of edges $\geq \frac{1}{2}kn$ since if $\deg(w) < k$ $(\{w\}, V \setminus \{w\})$ is smaller cut!

Contraction Algorithm Analysis

$$\Pr[\text{mistake in round 1}] \leq \frac{k}{\frac{1}{2}kn} = \frac{2}{n}$$

- Consider round $j + 1$:
  - Every cut in contracted graph is a cut in G, so every supernode has degree at least k.

$$\Pr[\text{mistake in } j+1 | \text{success so far}] \leq \frac{k}{\frac{1}{2}k(n-j)} = \frac{2}{n-j}$$

Final steps

- Let E_j be the event that (A, B) is not contracted in round j

$$\Pr[E_j | E_1 \cap \ldots E_{j-1}] \geq 1 - \frac{2}{n-j+1}$$

$$\begin{aligned}
&\Pr[E_1 \cap \ldots \cap E_{n-2}] \\
&= \Pr[E_1] \cdot \Pr[E_2|E_1] \cdot \ldots \Pr[E_{n-2}|E_1 \cap \ldots E_{n-3}] \\
&\geq \left(1 - \frac{2}{n}\right)\left(1 - \frac{2}{n-1}\right) \ldots \left(1 - \frac{2}{3}\right) \\
&= \frac{2}{n(n-1)}
\end{aligned}$$

Contraction Algorithm

Theorem. Alg finds global min cut with probability at least $1/\binom{n}{2}$.
Corollary. If we run $\binom{n}{2} \ln n$ times, alg succeeds with probability at least $1 - 1/n$.
Proof.

$$\Pr[\text{Fail all } t \text{ times}] \leq \left(1 - 1/\binom{n}{2}\right)^t$$

If $t = c\binom{n}{2}$ this is at most e^{-c}.

Global Min Cuts Takeaways

- Simple randomized algorithm works pretty well.
- Technical Tools
  - Chain Rule
  - Some calculus