COMPSCI 514: ALGORITHMS FOR DATA SCIENCE

Andrew McGregor

Lecture 1

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- The Large Synoptic Survey Telescope will take high definition photographs of the sky, producing 15 terabytes of data/night.
  - How do they denoise and compress the images? How do they detect anomalies such as changing brightness or position of objects to alert researchers?

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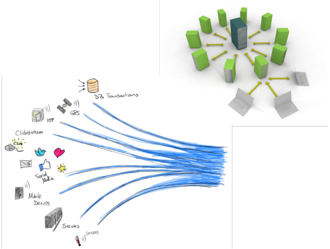

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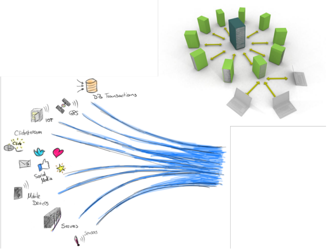

- Even 'simple' problems become very difficult in this setting.

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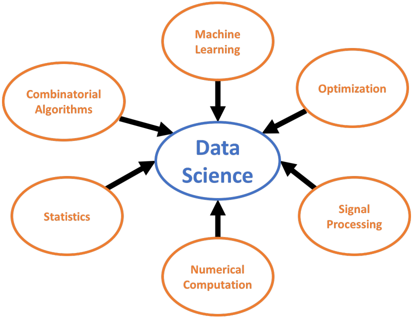
- How can Google estimate the number of unique search queries that are made in a given week? Given that no machine can store the full list of queries.
- When you use Shazam to identify a song from a recording, how does it provide an answer in < 10 seconds, without scanning over all ~ 8 million audio files in its database.

MOTIVATION FOR THIS CLASS

A Second Motivation: Data Science is highly interdisciplinary.



- Many techniques that aren't covered in the traditional CS algorithms curriculum.
- Emphasis on building comfort with mathematical tools that underly data science and machine learning.

WHAT WE'LL COVER

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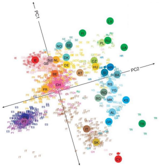
- Probability tools and concentration inequalities.
- Randomized hashing for efficient lookup, load balancing, and estimation. Bloom filters.
- Locality sensitive hashing and nearest neighbor search.
- Streaming algorithms: identifying frequent items in a data stream, counting distinct items, etc.
- Random compression of high-dimensional vectors: the Johnson-Lindenstrauss lemma, applications, and connections to the weirdness of high-dimensional geometry.

Section 2: Spectral Methods



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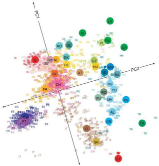

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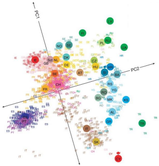
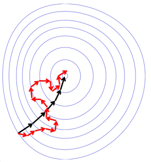


How do we identify the most important features of a dataset using linear algebraic techniques?

- Principal component analysis, low-rank approximation, dimensionality reduction.
- Singular value decomposition (SVD) and its applications to PCA, low-rank approximation, LSA, MDS, . . .
- Spectral graph theory. Spectral clustering, community detection, network visualization.
- Computing the SVD on large datasets via iterative methods.

Section 3: Optimization



Fundamental continuous optimization approaches that drive methods in machine learning and statistics.

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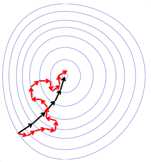

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- Stochastic and online gradient descent.
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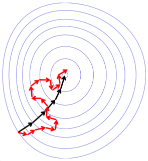

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A small taste of what you can find in COMPSCI 651.

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  - COMPSCI 589/689: Machine Learning

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For example: Bayes' rule in conditional probability. What it means for a vector x to be an eigenvector of a matrix A, projection, greedy algorithms, divide-and-conquer algorithms.

COURSE LOGISTICS

See course webpage for lecture slides and related readings:

https://people.cs.umass.edu/~mcgregor/CS514S23/

See Moodle page for this link if you lose it.

PERSONNEL

Professor: Andrew McGregor

TAs: Shib Dasgupta, Weronkia Nguyen, and Chenghao Lyu.

Together we've offer seven office hours, four in-person and three over Zoom. See Moodle page for locations, links, and times.

PIAZZA AND PARTICIPATION

We will use Piazza for class discussion and questions.

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You're helping yourself and others if you:

- Ask good clarifying questions and answering questions during lectures.
- Answer other students' or instructor questions on Piazza.
- Post helpful/interesting links on Piazza, e.g., resources covering class material, research articles related to class topics.

TEXTBOOKS AND MATERIALS

We will use material from two textbooks (links to free online versions on the course webpage): *Foundations of Data Science* and *Mining of Massive Datasets*, but will follow neither closely.

- I will sometimes post optional readings a few days prior to each class.
- Draft lecture notes will be posted before each class and potentially updated afterwards if necessary.

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Problem set submissions will be via Gradescope.

- See Moodle for a link to join.

WEEKLY QUIZZES

Will release a Moodle quiz most Fridays. It's due Monday at 8pm.

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- Designed as a check-in that you are following the material, and to help me make adjustments as needed.
- Should take under an hour per week. Open notes unless specified otherwise. Most questions easy but some more challenging ones.

GRADING

Grade Breakdown:

- Problem Sets: 25%. (One-time “lifeline extension” of 48 hours.)
- Weekly Quizzes: 20%. (No extensions but we’ll drop lowest quiz.)
- Midterm: 25%. Thursday October 20th
- Cumulative Final: 25%. (During Final’s Week.)
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Academic Honesty:

- A first violation cheating on a homework, quiz, or other assignment will result in a 0 on that assignment.
- A second violation, or cheating on an exam will result in failing the class.

DISABILITY SERVICES AND ACCOMMODATIONS

UMass is committed to making reasonable, effective, and appropriate accommodations to meet the needs to students with disabilities.

- If you have a documented disability **on file with Disability Services**, you may be eligible for reasonable accommodations in this course.
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I understand that people have different learning needs etc. If something isn't working for you in the class, please reach out and let's try to work it out.

Questions?

Section 1: Randomized Methods & Sketching

SOME PROBABILITY REVIEW

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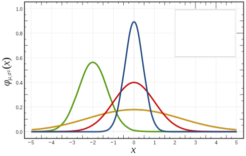

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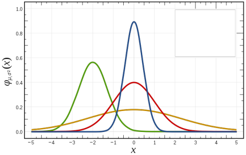

For any scalar α, $\mathbb{E}[\alpha \cdot \mathbf{X}] = \alpha \cdot \mathbb{E}[\mathbf{X}]$ and $\text{Var}[\alpha \cdot \mathbf{X}] = \alpha^2 \cdot \text{Var}[\mathbf{X}]$.

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Using the definition of conditional probability, independence means:

$$\frac{\Pr(A \cap B)}{\Pr(B)} = \Pr(A) \implies \Pr(A \cap B) = \Pr(A) \cdot \Pr(B).$$

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Example: What is the probability that for two independent dice rolls the first is odd and the second is odd?

$$\begin{aligned}\Pr(D_1 \in \{1,3,5\} \cap D_2 \in \{1,3,5\}) &= \Pr(D_1 \in \{1,3,5\}) \cdot \Pr(D_2 \in \{1,3,5\}) \\ &= 1/4\end{aligned}$$

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$$\begin{aligned}&\Pr(D_1 = 1 \cap D_1 + D_2 \text{ is odd}) \\ &= \Pr(D_1 = 1) \cdot \Pr(D_1 + D_2 \in \{1,3,5,7,9,11\} | D_1 = 1) \\ &= 1/6 \cdot \Pr(D_2 \in \{2,4,6\}) = 1/6 \cdot 1/2 = \Pr(D_1 = 1) \cdot \Pr(D_1 + D_2 \text{ is odd})\end{aligned}$$

LINEARITY OF EXPECTATION AND VARIANCE

When are the expectation and variance linear?

I.e., under what conditions on $\mathbf{X}$ and $\mathbf{Y}$ do we have:

$$\mathbb{E}[\mathbf{X} + \mathbf{Y}] = \mathbb{E}[\mathbf{X}] + \mathbb{E}[\mathbf{Y}]$$

and

$$\text{Var}[\mathbf{X} + \mathbf{Y}] = \text{Var}[\mathbf{X}] + \text{Var}[\mathbf{Y}].$$

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&= \sum_{s\in S} s\sum_{t\in T} \Pr(\mathbf{X}=s\cap\mathbf{Y}=t) + \sum_{t\in T} t\sum_{s\in S} \Pr(\mathbf{X}=s\cap\mathbf{Y}=t)
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&= \sum_{s\in S}\sum_{t\in T}\Pr(\mathbf{X}=s\cap\mathbf{Y}=t)s + \sum_{s\in S}\sum_{t\in T}\Pr(\mathbf{X}=s\cap\mathbf{Y}=t)t\\
&= \sum_{s\in S}s\sum_{t\in T}\Pr(\mathbf{X}=s\cap\mathbf{Y}=t) + \sum_{t\in T}t\sum_{s\in S}\Pr(\mathbf{X}=s\cap\mathbf{Y}=t)
\end{aligned}$$

$\mathbb{E}[\mathbf{X}+\mathbf{Y}] = \mathbb{E}[\mathbf{X}] + \mathbb{E}[\mathbf{Y}]$ for any random variables $\mathbf{X}$ and $\mathbf{Y}$.

Proof:

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&= \sum_{s\in S}s\Pr(\mathbf{X}=s) + \sum_{t\in T}t\Pr(\mathbf{Y}=t)
\end{aligned}$$

(law of total probability)

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&= \sum_{s\in S} s\Pr(\mathbf{X}=s) + \sum_{t\in T} t\Pr(\mathbf{Y}=t) \\
&\qquad\qquad\qquad\qquad\qquad\qquad \text{(law of total probability)} \\
&= \mathbb{E}[\mathbf{X}] + \mathbb{E}[\mathbf{Y}].
\end{aligned}$$

LINEARITY OF VARIANCE

$$\text{Var}[\mathbf{X} + \mathbf{Y}] = \text{Var}[\mathbf{X}] + \text{Var}[\mathbf{Y}]$$

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(linearity of expectation)

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&\qquad\qquad\qquad\qquad\qquad\qquad \text{(linearity of expectation)} \\
&= \mathbb{E}[\mathbf{X}^2] + 2\mathbb{E}[\mathbf{XY}] + \mathbb{E}[\mathbf{Y}^2] - \mathbb{E}[\mathbf{X}]^2 - 2\mathbb{E}[\mathbf{X}] \cdot \mathbb{E}[\mathbf{Y}] - \mathbb{E}[\mathbf{Y}]^2
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&= \mathbb{E}[\mathbf{X}^2] + \mathbb{E}[\mathbf{Y}^2] - \mathbb{E}[\mathbf{X}]^2 - \mathbb{E}[\mathbf{Y}]^2
\end{aligned}$$

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&= \mathbb{E}[\mathbf{X}^2] + \mathbb{E}[\mathbf{Y}^2] - \mathbb{E}[\mathbf{X}]^2 - \mathbb{E}[\mathbf{Y}]^2 \\
&= \text{Var}[\mathbf{X}] + \text{Var}[\mathbf{Y}].
\end{aligned}$$