# COMPSCI 514: ALGORITHMS FOR DATA SCIENCE

Andrew McGregor

Lecture 12

# CENTRAL LIMIT THEOREM

# INTERPRETATION AS A CENTRAL LIMIT THEOREM

**Bernstein Inequality (Simplified):** Consider independent random variables $\mathbf{X}_1, \ldots, \mathbf{X}_n$ falling in [-1,1]. Let $\mu = \mathbb{E}[\sum \mathbf{X}_i]$, $\sigma^2 = \text{Var}[\sum \mathbf{X}_i]$, and $s \leq \sigma$. Then:

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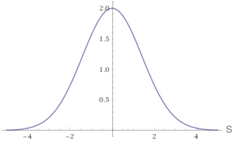

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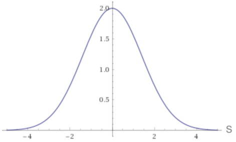


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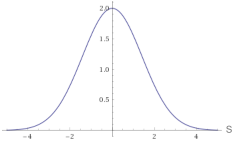


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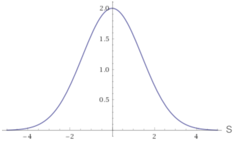


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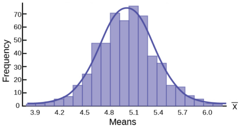
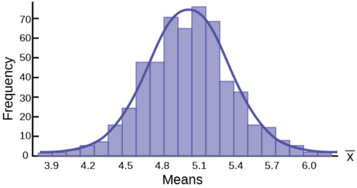
**Central Limit Theorem Interpretation:** Bernstein's inequality gives a quantitative version of the CLT. The distribution of the sum of *bounded* independent random variables can be upper bounded with a Gaussian (normal) distribution.



**Stronger Central Limit Theorem:** The distribution of the sum of *n* *bounded* independent random variables converges to a Gaussian (normal) distribution as *n* goes to infinity.



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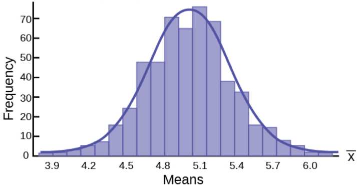


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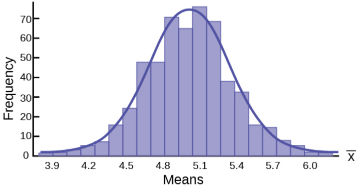


- Why is the Gaussian distribution is so important in statistics, science, ML, etc.?
- Many random variables can be approximated as the sum of a large number of small and roughly independent random effects. Thus, their distribution looks Gaussian by CLT.

# SUMMARY OF FIRST SECTION

# WHAT WE'VE COVERED

- **Probability Tools:** Linearity of Expectation, Linear of Variance of Independent Variables, Concentration Bounds (Markov, Chebyshev, Bernstein, Chernoff), Union Bound, Median Trick.
- **Hash Tables and Bloom Filters:** Analyzing collisions. Building 2-level hash tables. Bloom filters and false positive rates.
- **Locality Sensitive Hashing:** MinHash for Jaccard Similarity, SimHash for Cosine Similarity. Nearest Neighbor. All-Pairs Similarity Search.
- **Small Space Data Stream Algorithms:** a) distinct items, b) frequent elements, c) frequent moments (homework).
- **Johnson Lindenstrauss Lemma:** Reducing dimension of vectors via random projection such that pairwise distances are approximately preserved. Application to clustering.

# RANDOMIZED ALGORITHMS UNIT TAKEAWAYS

- Randomization is an important tool in working with large datasets.
- Lets us solve 'easy' problems that get really difficult on massive datasets. Fast/space efficient look up (hash tables and bloom filters), distinct items counting, frequent items counting, near neighbor search (LSH), etc.
- The analysis of randomized algorithms sometimes leads to complex output distributions, which we can't compute exactly. We use concentration inequalities to bound these distributions and behaviors like accuracy, space usage, and runtime.
- Concentration inequalities and probability tools used in randomized algorithms are also fundamental in statistics, machine learning theory, probabilistic modeling of complex systems, etc.

# USEFUL PROBABILITY FACTS (1/2)

- Linearity of Expectation: For any random variables $X_1, \ldots, X_n$ and constants $c_1, \ldots, c_n$,

$$\mathbb{E}[c_1 X_1 + \ldots + c_n X_n] = c_1 \mathbb{E}[X_1] + \ldots + c_n \mathbb{E}[X_n]$$

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- Independent Random Variables: $X_1, X_2, \ldots X_n$ are independent random variables if for any set $S \subset [n]$ and values $a_1, a_2, \ldots, a_n$

$$\Pr(X_i = a_i \text{ for all } i \in S) = \prod_{i \in S} \Pr(X_i = a_i) \ .$$

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- Linearity of Variance: If $X_1, \ldots, X_n$ are independent (in fact 2-wise independent suffices) then for any constants $c_1, \ldots, c_n$

$$\operatorname{Var}[c_1X_1 + \ldots + c_nX_n] = c_1^2 \operatorname{Var}[X_1] + \ldots + c_n^2 \operatorname{Var}[X_n]$$

# USEFUL PROBABILITY FACTS (2/2)

- Union Bound: For any events $A_1, A_2, A_3, \ldots$

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- If $Y = X_1 + \ldots + X_n$ where each $X_i$ are independent and $p = \Pr[X_1 = 1] = \ldots = \Pr[X_n = 1]$ then $Y$ is a binomial random variable. Using linearity of expectation and variance,

$$\mathbb{E}[Y] = np \qquad \mathrm{Var}[Y] = np(1-p)$$

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$$\Pr[\text{collisions}] = \Pr[\max(R_1, \ldots, R_m) > 1] \leq 1/8 \text{ if } m > 4n^2$$

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- In the exam, you'll be expected to do calculations like these.

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- The probability the next $k$ balls thrown all land in non-empty bins is

$$(1 - T/m)^k$$

and this lets us analyze the false positive rate of a Bloom filter.

- Hash function $\mathbf{h} : U \rightarrow [n]$ is two universal if:

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- Hash function $\mathbf{h} : U \to [n]$ is fully independent if $\{h(e)\}_{e \in U}$ are independent and each $h(e)$ is uniform in $[n]$.

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- Bernstein generalizes Chernoff to arbitrary bounded $X_i$ variables.

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- Median Trick: Let $t = t_1 t_2$ where $t_1 = \frac{4\sigma^2}{\epsilon^2 q^2}$ and $t_2 = O(\log \frac{1}{\delta})$. Let $A_1$ be average of first $t_1$ results, let $A_2$ be average of next $t_1$ results etc. Then,

$$\Pr[|A_i - q| \geq \epsilon q] \leq 1/4$$

and $\Pr[|\text{median}(A_1, \ldots, A_{t_2}) - q| \geq \epsilon q] \leq \delta$.

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- Bloom Filter:
  - Does not actually store the items in $S$, just a binary array from which we make various deductions.

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- Also saw stacked hash tables in the homework.

- Designed a hash function for hashing sets such that for sets $A$ and $B$, $\Pr[MH(A) = MH(B)] = J(A, B) = \frac{|A \cap B|}{|A \cup B|}$.

$$MH(A) = \min_{x \in A} h(x) \quad \text{where} \quad h : U \to [0, 1] \text{ is fully independent}$$

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- To find all pairs of similar sets amongst $A_1, A_2, A_3, \ldots$ only compare a pair if there exists $i$, their $i$th signatures match.

# DATA STREAMS ALGORITHMS

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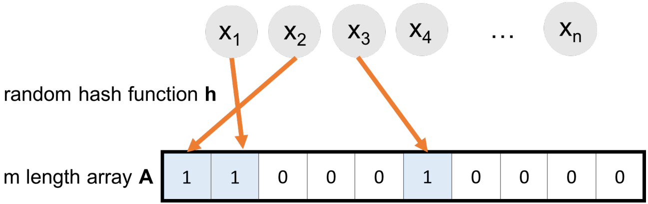
  - Frequently Elements Items: Can return a set $S$ such that:

    $$f_i \geq m/k \text{ implies } i \in S \qquad \text{and} \qquad i \in S \text{ implies } f_i \geq m(1 - \epsilon)/k$$

    with probability $1 - \delta$ in $O(k/\epsilon \cdot \log 1/\delta)$ space.
  - Sum of Powers: In the homework we considered estimating quantities such as $\sum f_i^k$.

Count-Min Sketch: A random hashing based method closely related to bloom filters.



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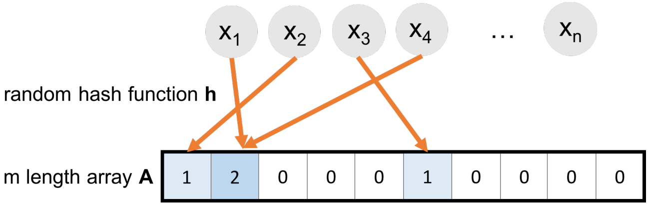

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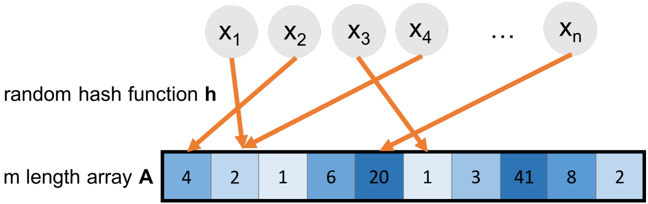

# FREQUENT ELEMENTS WITH COUNT-MIN SKETCH

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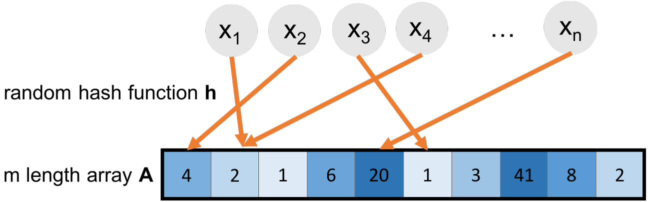


Use $A[\mathbf{h}(x)]$ to estimate $f(x)$, the frequency of $x$ in the stream.

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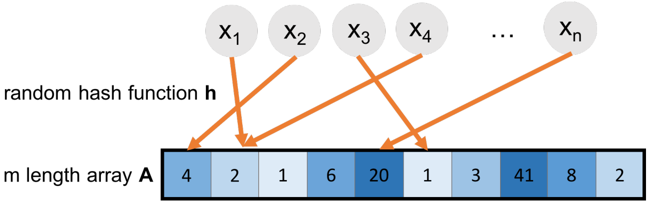


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How can we increase this probability to $1 - \delta$ for arbitrary $\delta > 0$?

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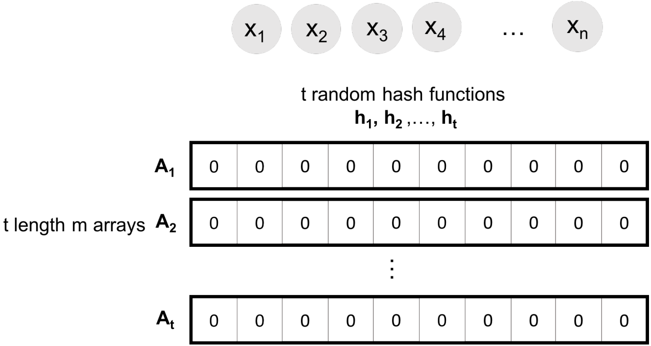

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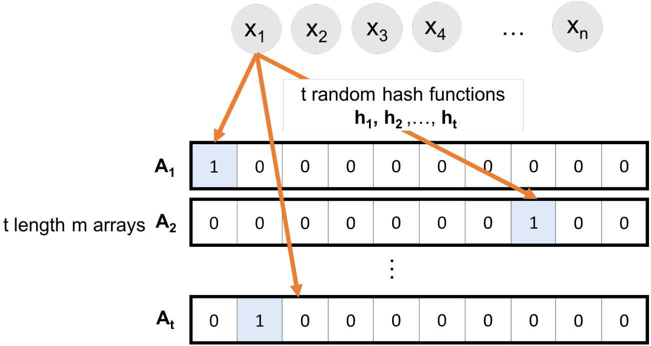

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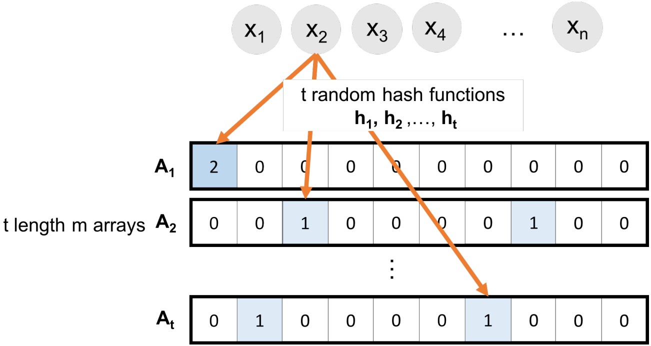

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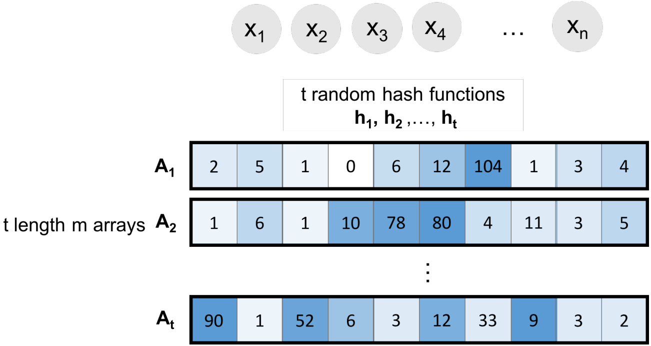

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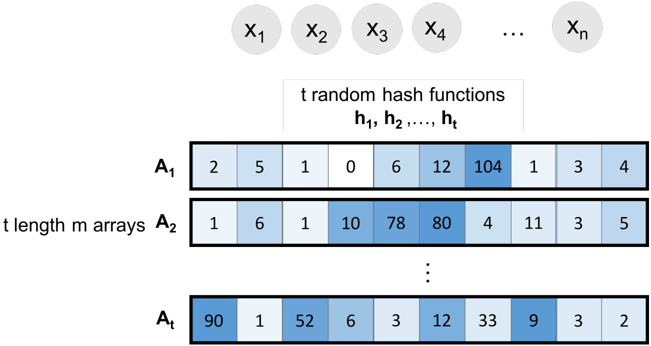

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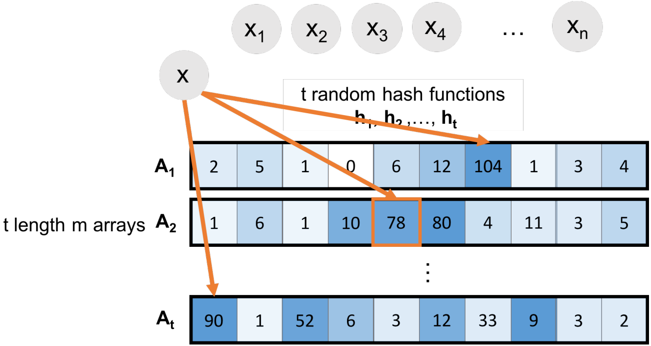

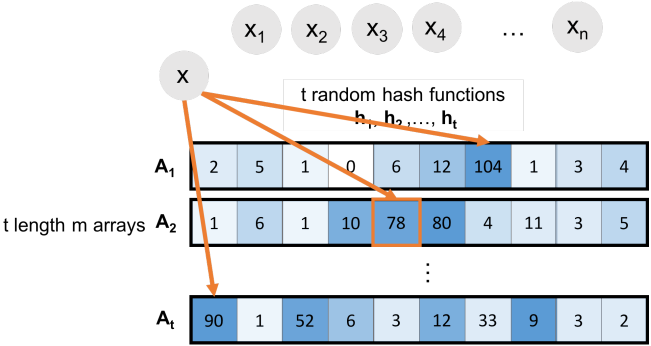


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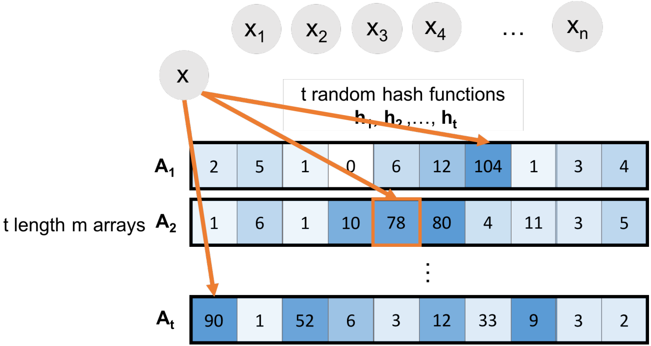


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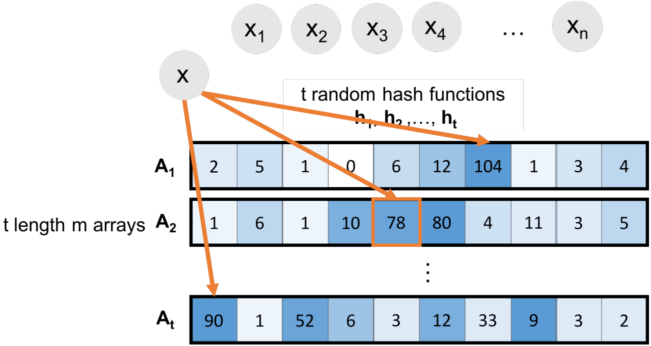


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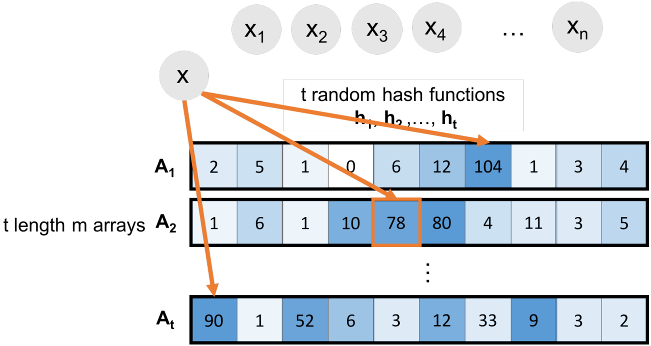


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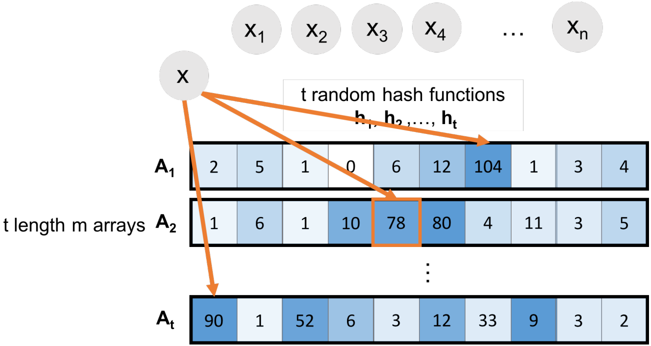


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- Setting $m = 2k/\epsilon$ ensures $2n/m = \epsilon n/k$ and that's enough to determine whether we need to output the element.

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The more distinct hashes we see, the higher we expect this maximum to be.

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Flajolet-Martin (LogLog) algorithm and HyperLogLog.

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With $d$ distinct elements, roughly what do we expect **m** to be?

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**Note:** Careful averaging of estimates from multiple hash functions.

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- Set the maximum # of trailing zeros to the maximum in the two sketches.

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