COMPSCI 514: ALGORITHMS FOR DATA SCIENCE

Andrew McGregor

Lecture 13

Next Few Classes: Low-rank approximation, the SVD, and principal component analysis (PCA).

- Reduce d-dimensional data points to a smaller dimension m.
- Like JL, compression is linear, i.e., by applying a matrix.
- Chose matrix taking into account structure of dataset.
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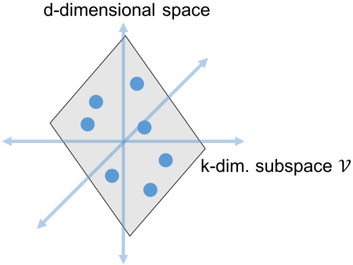
- A set of vectors $\mathcal{B}$ is a basis for a set of vectors $\mathcal{V}$, if every vector in $\mathcal{V}$ is a linear combination of vectors in $\mathcal{B}$.
- The dimension of $\mathcal{V}$ is the size of its smallest basis.

Assume that data points $\vec{x}_1, \ldots, \vec{x}_n$ lie in some k-dimensional subspace $\mathcal{V}$ of $\mathbb{R}^d$.



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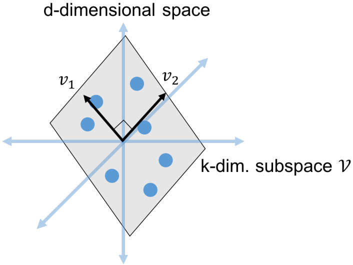

Claim: Let $\vec{v}_1, \ldots, \vec{v}_k$ be an orthonormal basis for $\mathcal{V}$ and $\mathbf{V} \in \mathbb{R}^{d \times k}$ be the matrix with these vectors as its columns. For all $\vec{x}_i, \vec{x}_j$:

$$\|\mathbf{V}^T \vec{x}_i - \mathbf{V}^T \vec{x}_j\|_2 = \|\vec{x}_i - \vec{x}_j\|_2.$$

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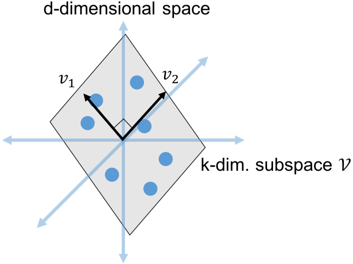

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$$\|\mathbf{V}^T \vec{x}_i - \mathbf{V}^T \vec{x}_j\|_2 = \|\vec{x}_i - \vec{x}_j\|_2.$$

That is, $\mathbf{V}^T \in \mathbb{R}^{k \times d}$ is a linear embedding of $\vec{x}_1, \ldots, \vec{x}_n$ into k dimensions with no distortion.

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- So $\|\vec{y}\|_2^2 = \vec{c}^T \vec{c} = \|\mathbf{V}^T \vec{y}\|_2^2$.

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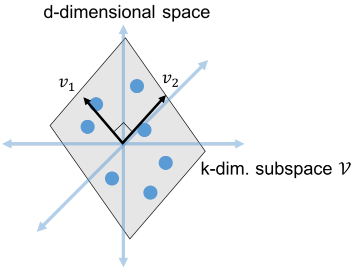

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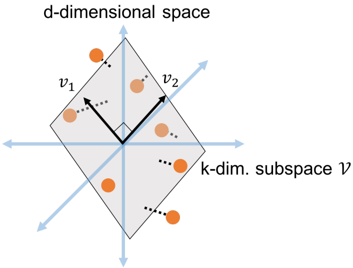

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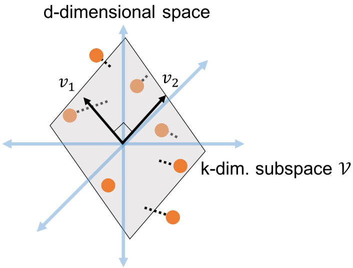

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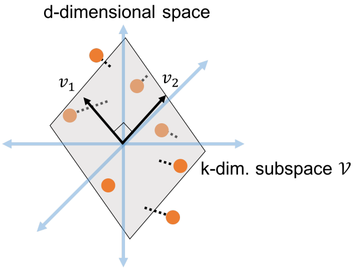

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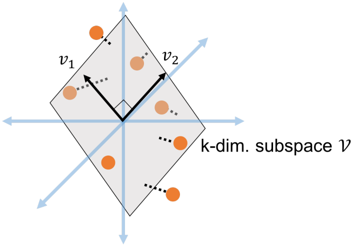

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- How do we find $\mathcal{V}$ and $\mathbf{V}$?
- How good is the embedding?

- Every data point $\vec{x}_i$ (row of $\mathbf{X}$) can be written as

$$\vec{x}_i = \mathbf{V}\vec{c}_i = c_{i,1} \cdot \vec{v}_1 + \ldots + c_{i,k} \cdot \vec{v}_k$$

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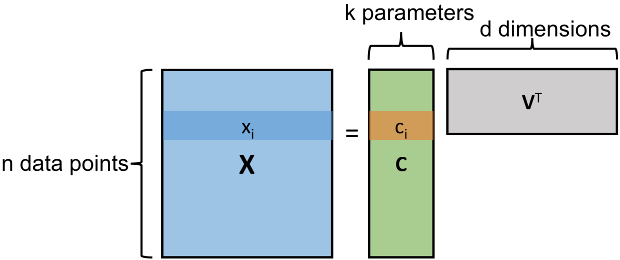

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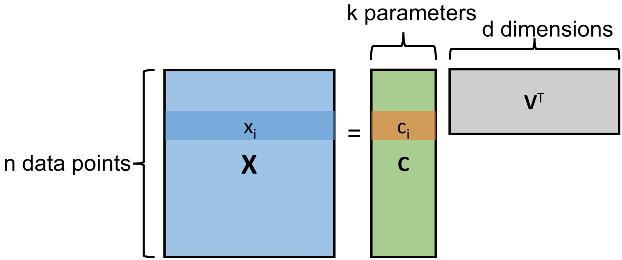

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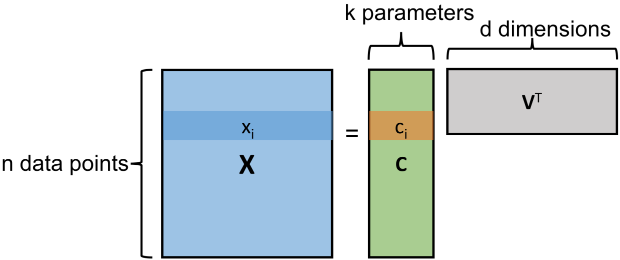
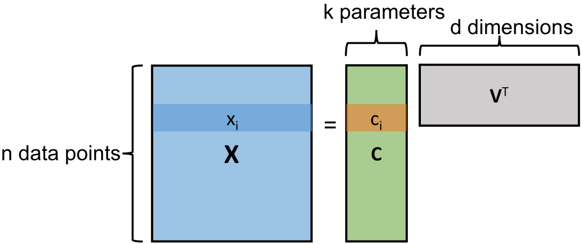


- $\mathbf{X}$ can be represented by $(n+d) \cdot k$ parameters vs. $n \cdot d$.
- The rows of $\mathbf{X}$ are spanned by k vectors: the columns of $\mathbf{V}$ $\implies$ the columns of $\mathbf{X}$ are spanned by k vectors: the columns of $\mathbf{C}$.

LOW-RANK FACTORIZATION

Claim: If $\vec{x}_1, \ldots, \vec{x}_n$ lie in a k-dimensional subspace with orthonormal basis $\mathbf{V} \in \mathbb{R}^{d \times k}$, the data matrix can be written as $\mathbf{X} = \mathbf{C}\mathbf{V}^T$.



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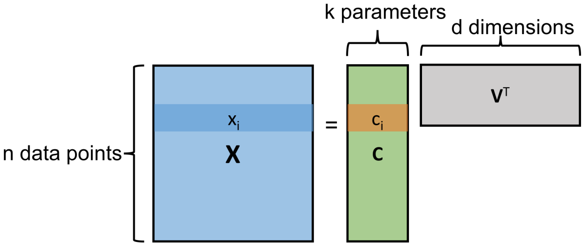

Exercise: What is this coefficient matrix C? **Hint:** Use that $\mathbf{V}^T\mathbf{V} = \mathbf{I}$.

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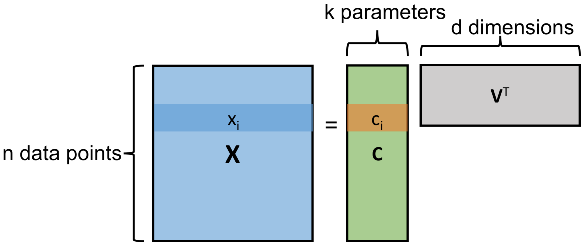

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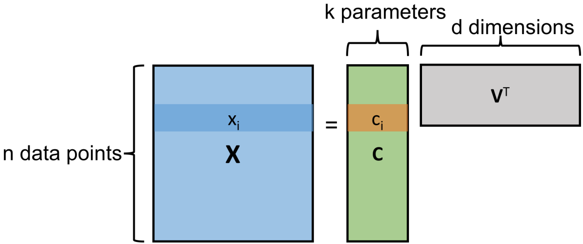

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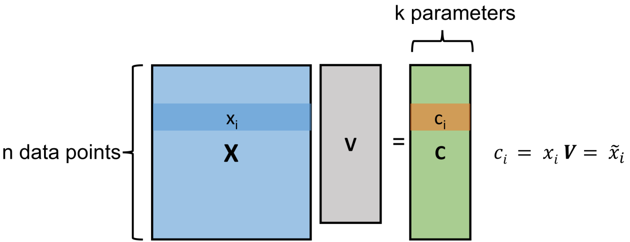

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$\vec{x}_1, \ldots, \vec{x}_n \in \mathbb{R}^d$: data points, $\mathbf{X} \in \mathbb{R}^{n \times d}$: data matrix, $\vec{v}_1, \ldots, \vec{v}_k \in \mathbb{R}^d$: orthogonal basis for subspace $\mathcal{V}$. $\mathbf{V} \in \mathbb{R}^{d \times k}$: matrix with columns $\vec{v}_1, \ldots, \vec{v}_k$.

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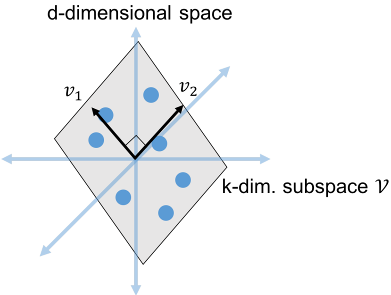

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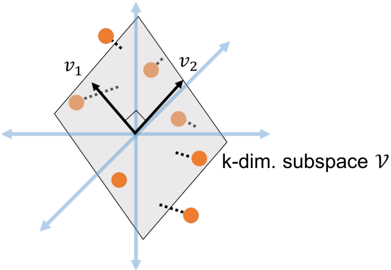

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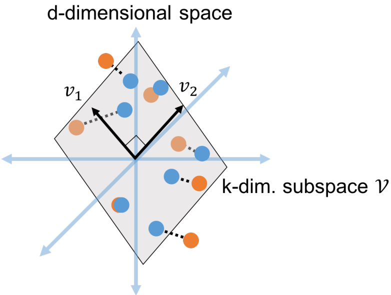

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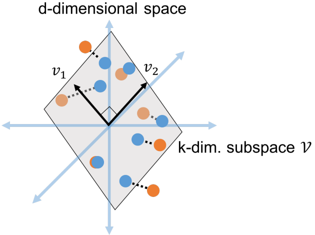

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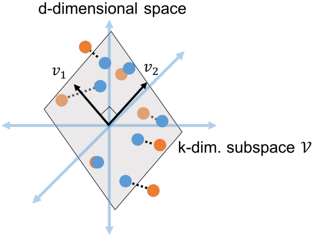

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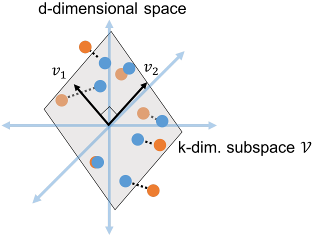

Note: $\mathbf{X}\mathbf{V}\mathbf{V}^T$ has rank k. It is a low-rank approximation of $\mathbf{X}$. Later we'll show that:

$$\mathbf{X}\mathbf{V}\mathbf{V}^{\mathbf{T}} = \underset{\mathbf{B} \text{ with rows in } \mathcal{V}}{\arg\min} \|\mathbf{X} - \mathbf{B}\|_F^2$$

where $\|\mathbf{A}\|_F$ is defined as $\sqrt{\sum_{i,j} \mathbf{A}_{i,j}^2}$.

LOW-RANK APPROXIMATION

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Next Time: How to find the subspace $\mathcal{V}$ and correspondingly $\mathbf{V}$.

a step back: why low-rank approximation?

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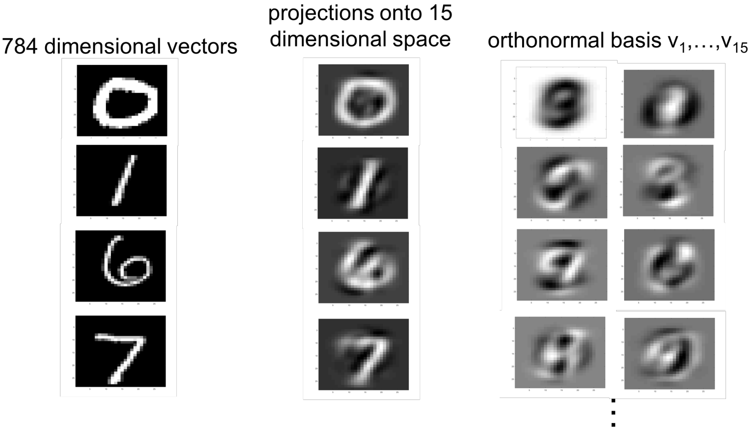

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