COMPSCI 514: ALGORITHMS FOR DATA SCIENCE

Andrew McGregor

Lecture 16

Last Class: Low-Rank Approximation, Eigendecomposition, PCA

- For any symmetric square matrix A, we can write $\mathbf{A} = \mathbf{V}\mathbf{\Lambda}\mathbf{V}^T$ where columns of $\mathbf{V}$ are orthonormal eigenvectors.
- Can approximate data lying close to in a k-dimensional subspace by projecting data points into that space.
- Can find the best k-dimensional subspace via eigendecomposition applied to $\mathbf{X}^T\mathbf{X}$ (PCA).
- Measuring error in terms of the eigenvalue spectrum.

This Class: SVD and Applications

- SVD and connection to eigenvalue value decomposition.
- Applications of low-rank approximation beyond compression.

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- $\mathbf{U}$ has orthonormal columns $\vec{u}_1, \ldots, \vec{u}_r \in \mathbb{R}^n$ (left singular vectors).
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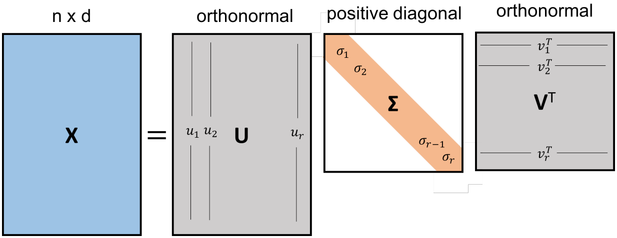

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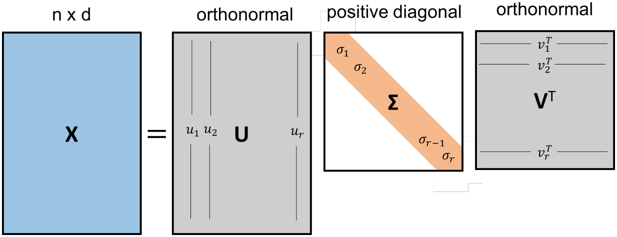

The 'swiss army knife' of modern linear algebra.

Writing $\mathbf{X} \in \mathbb{R}^{n \times d}$ in its singular value decomposition $\mathbf{X} = \mathbf{U}\boldsymbol{\Sigma}\mathbf{V}^T$:

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Exercise: $\mathbf{U}_k\mathbf{U}_k^T\mathbf{X} = \mathbf{X}\mathbf{V}_k\mathbf{V}_k^T = \mathbf{U}_k\mathbf{\Sigma}_k\mathbf{V}_k^T$

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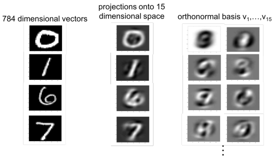
Row (data point) compression



Column (feature) compression

10000* bathrooms+ 10* (sq. ft.) ≈ list price

	bedrooms	bathrooms	sq.ft.	floors	list price	sale price
home 1	2	2	1800	2	200,000	195,000
home 2	4	2.5	2700	1	300,000	310,000
⋮	⋮	⋮	⋮	⋮	⋮	⋮
home n	5	3.5	3600	3	450,000	450,000

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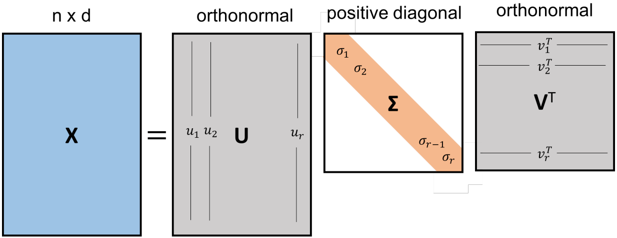

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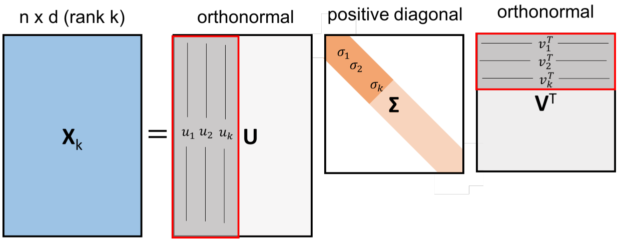

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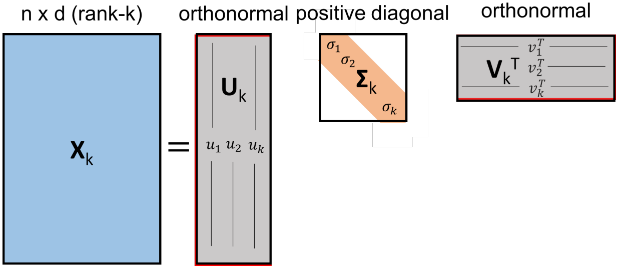

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- This establishes that $\mathbf{XV} = \mathbf{U\Sigma}$ and that $\mathbf{V}$ and $\mathbf{U}$ have the required properties.
- To see rest of the details, see `https://math.mit.edu/classes/18.095/2016IAP/lec2/SVD_Notes.pdf`

APPLICATIONS OF LOW-RANK APPROXIMATION

Rest of Class: Examples of how low-rank approximation is applied in a variety of data science applications.

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- Used for many reasons other than dimensionality reduction/data compression.

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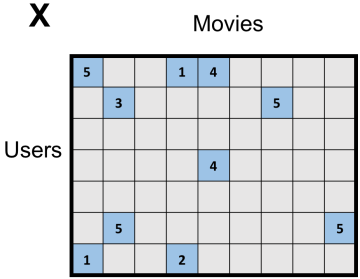

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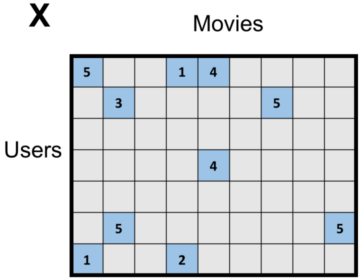

Solve: $\boldsymbol{Y} = \underset{\text{rank }-k \ \mathbf{B}}{\arg\min} \sum_{\text{observed } (j,k)} [\mathbf{X}_{j,k} - \mathbf{B}_{j,k}]^2$

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Y

Movies

Users

4.9	3.1	3	**1.1**	**3.8**	4.1	4.1	3.4	4.6
3.6	**3**	3	1.2	3.8	4.2	**5**	3.4	4.8
2.8	3	3	2.3	3	3	3	3	3.2
3.4	3	3	4	**4.1**	4.1	4.2	3	3
2.8	3	3	2.3	3	3	3	3	3.4
2.2	**5**	3	4	4.2	3.9	4.4	4	**5.3**
1	3.3	3	**2.2**	3.1	2.9	3.2	1.5	1.8

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Under certain assumptions, can show that **Y** well approximates **X** on both the observed and (most importantly) unobserved entries.

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ENTITY EMBEDDINGS

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- Documents (for topic-based search and classification)
- Words (to identify synonyms, translations, etc.)
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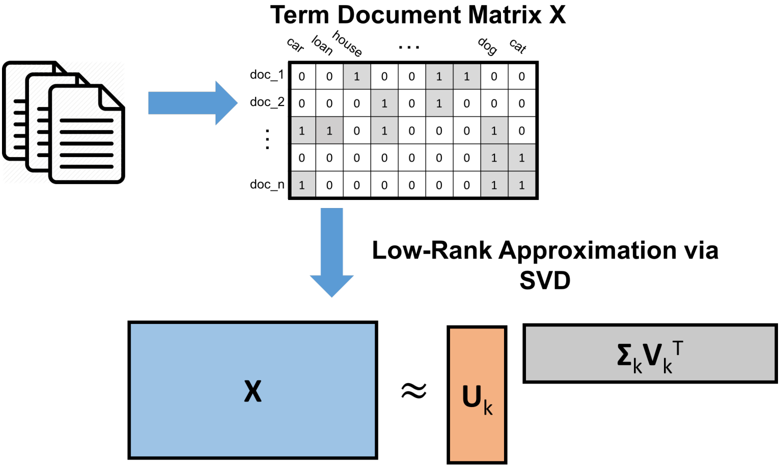
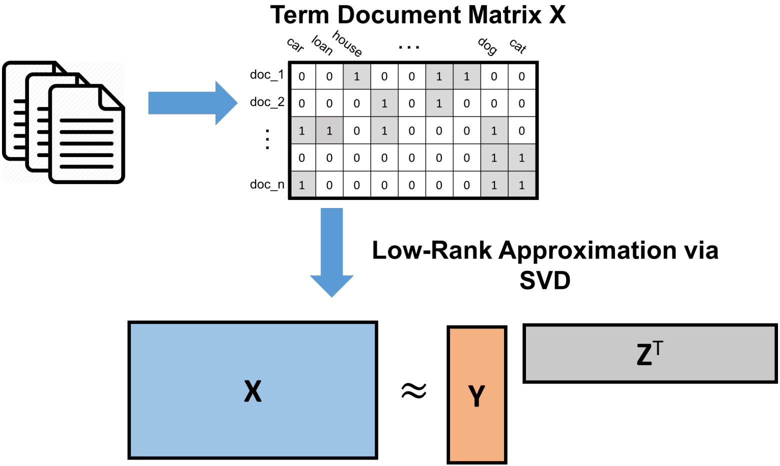
Usual Approach: Convert each item into a high-dimensional feature vector and then apply low-rank approximation.



	car	loan	house	...				dog	cat
doc_1	0	0	1	0	0	1	1	0	0
doc_2	0	0	0	1	0	1	0	0	0
⋮	1	1	0	1	0	0	0	1	0
	0	0	0	0	0	0	0	1	1
doc_n	1	0	0	0	0	0	0	1	1



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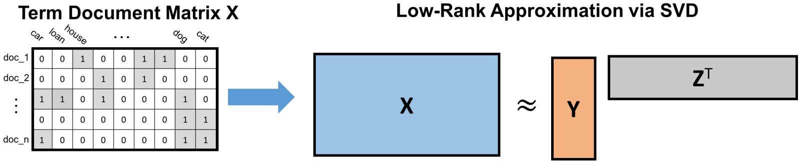
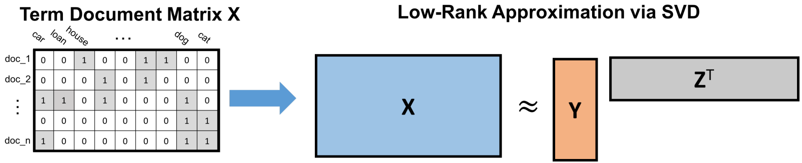
doc_1	0	0	1	0	0	1	1	0	0
doc_2	0	0	0	1	0	1	0	0	0
⋮	1	1	0	1	0	0	0	1	0
	0	0	0	0	0	0	0	1	1
doc_n	1	0	0	0	0	0	0	1	1

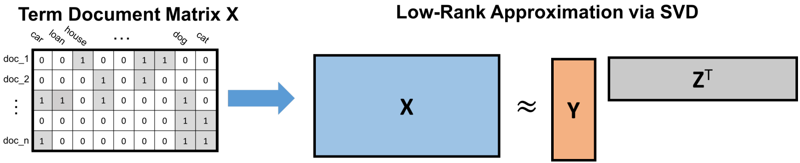
EXAMPLE: LATENT SEMANTIC ANALYSIS





- If the error $\|\mathbf{X} - \mathbf{Y}\mathbf{Z}^T\|_F$ is small, then on average,

$$\mathbf{X}_{i,a} \approx (\mathbf{Y}\mathbf{Z}^T)_{i,a} = \langle \vec{y}_i, \vec{z}_a \rangle.$$



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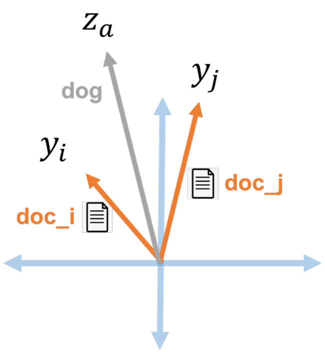
- I.e., $\langle \vec{y}_i, \vec{z}_a \rangle \approx 1$ when doc_i contains $word_a$.

EXAMPLE: LATENT SEMANTIC ANALYSIS

If doc_i and doc_j both contain $word_a$, $\langle \vec{y}_i, \vec{z}_a \rangle \approx \langle \vec{y}_j, \vec{z}_a \rangle \approx 1$ If doc_i and doc_j both don't contain $word_a$, $\langle \vec{y}_i, \vec{z}_a \rangle \approx \langle \vec{y}_j, \vec{z}_a \rangle \approx 0$



Since this applies for all words, documents with that involve a similar set of words tend to have high dot product with each other.

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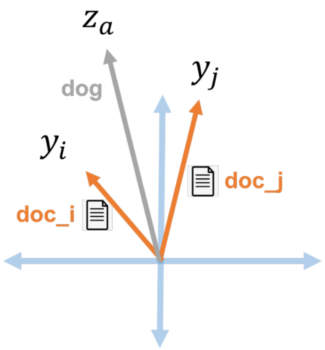

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Another View: Column of **Y** represent 'topics'. $\vec{y}_i(j)$ indicates how much doc_i belongs to topic j. $\vec{z}_a(j)$ indicates how much $word_a$ associates with that topic.

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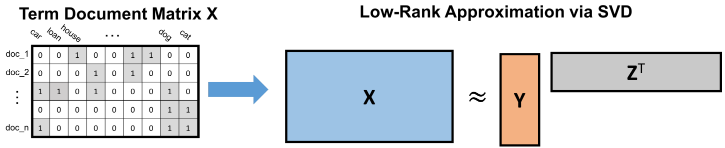

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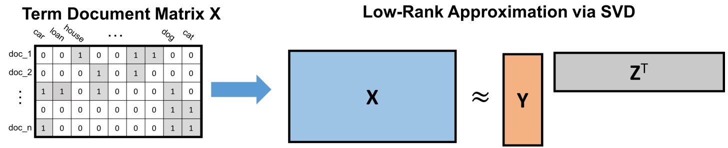

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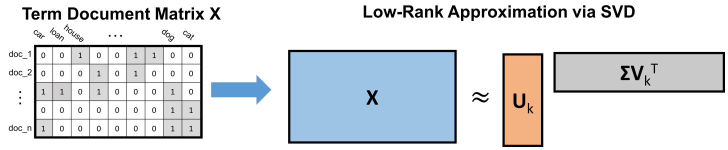

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EXAMPLE: WORD EMBEDDING

LSA gives a way of embedding words into k-dimensional space.

- Embedding is via low-rank approximation of $\mathbf{X}^T\mathbf{X}$: where $(\mathbf{X}^T\mathbf{X})_{a,b}$ is the number of documents that both $word_a$ and $word_b$ appear in.

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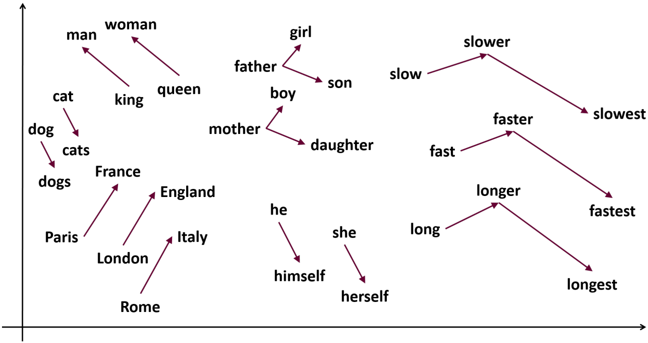
- Many ways to measure similarity: number of sentences both occur in, number of times both appear in the same window of w words, in similar positions of documents in different languages, etc.
- Replacing $\mathbf{X}^T\mathbf{X}$ with these different metrics (sometimes appropriately transformed) leads to popular word embedding algorithms: word2vec, GloVe, fastText, etc.


man
woman
king
queen
cat
cats
dog
dogs
France
Paris
England
London
Italy
Rome
father
girl
son
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boy
daughter
he
himself
she
herself
slow
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slowest
fast
faster
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long
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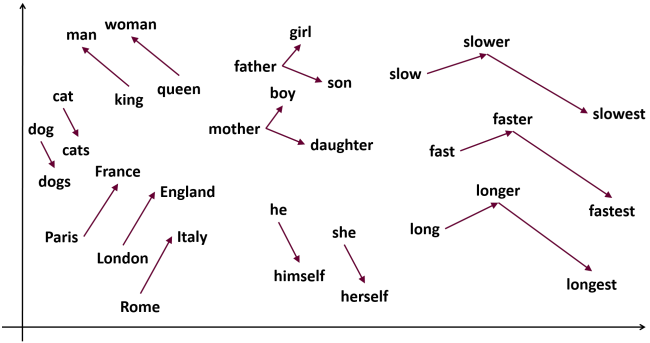
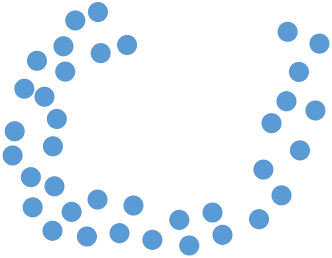
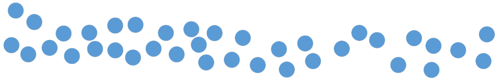


Note: word2vec is typically described as a neural-network method, but it is really just low-rank approximation of a specific similarity matrix. *Neural word embedding as implicit matrix factorization*, Levy and Goldberg.

GRAPH EMBEDDINGS



Is this set of points compressible? Does it lie close to a low-dimensional subspace? (A 1-dimensional subspace of $\mathbb{R}^d$.)



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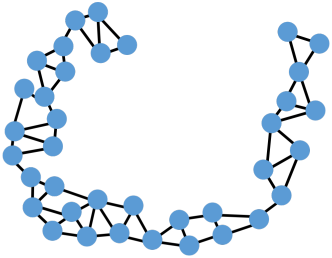

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A common way of automatically identifying this non-linear structure is to connect data points in a graph. E.g., a k-nearest neighbor graph.

- Connect items to similar items, possibly with higher weight edges when they are more similar.

LINEAR ALGEBRAIC REPRESENTATION OF A GRAPH

Once we have connected n data points $x_1, \ldots, x_n$ into a graph, we can represent that graph by its (weighted) adjacency matrix.

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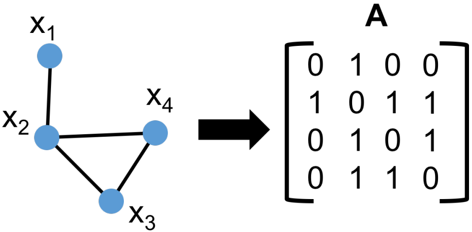

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ADJACENCY MATRIX EIGENVECTORS

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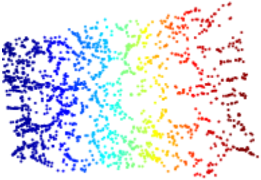
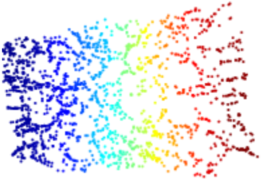
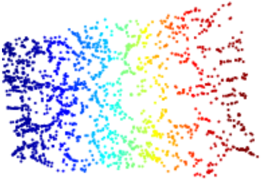
  where we showed the equality in Lecture 14.





Step 1: Produce a nearest neighbor graph based on your input data in $\mathbb{R}^d$.



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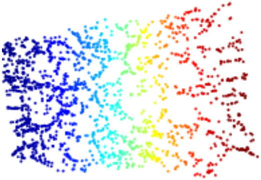

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Step 3: Work with the data in the embedded space. Where distances approximate distances in your original 'non-linear space.'