COMPSCI 514: ALGORITHMS FOR DATA SCIENCE

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Lecture 2

Today:

- Investigate linearity of expectation and variance.
- Algorithmic application of linearity of expectation and variance.
- Introduce Markov's inequality, a fundamental concentration bound, that let us prove that a random variable lies close to its expectation with good probability.
- Learn about random hash functions, which are a key tool in randomized methods for data processing. Probabilistic analysis via linearity of expectation.

SOME PROBABILITY REVIEW

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$$\text{Var}[\mathbf{X}] = \mathbb{E}[(\mathbf{X} - \mathbb{E}[\mathbf{X}])^2].$$
- Two random variables $\mathbf{X}$, $\mathbf{Y}$ are **independent** if for all s, t, $\{\mathbf{X} = s\}$ and $\{\mathbf{Y} = t\}$ are independent events. In other words:
$$\Pr(\{\mathbf{X} = s\} \cap \{\mathbf{Y} = t\}) = \Pr(\mathbf{X} = s) \cdot \Pr(\mathbf{Y} = t).$$

LINEARITY OF EXPECTATION AND VARIANCE

When are the expectation and variance linear?

I.e., under what conditions on $\mathbf{X}$ and $\mathbf{Y}$ do we have:

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Last time we showed that linearity of expectation is true regardless of whether the random variables were independent.

$\mathbf{X}, \mathbf{Y}$: any two random variables.

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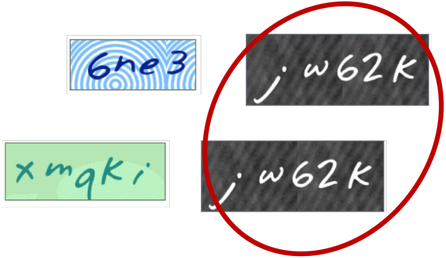
- You want to independently verify this claimed database size.
- You could make test checks until you see $1,000,000$ unique CAPTCHAS: would take $\geq 1,000,000$ checks!

An Idea: You run some test security checks and see if any duplicate CAPTCHAS show up. If you're seeing duplicates after not too many checks, the database size is probably not too big.



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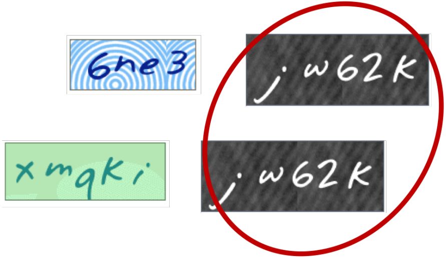

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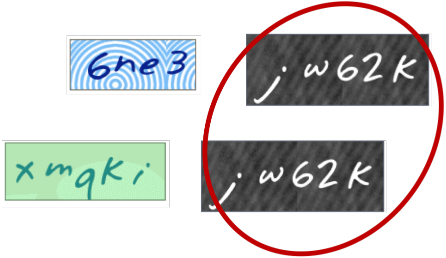

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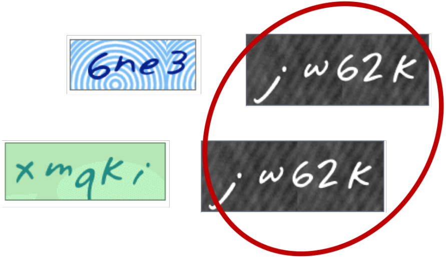

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Note that if the same CAPTCHA shows up four times this counts as $\binom{4}{2}$ duplicates.

Let $\mathbf{D}_{i,j} = 1$ if tests i and j give the same CAPTCHA, and 0 otherwise. An indicator random variable.

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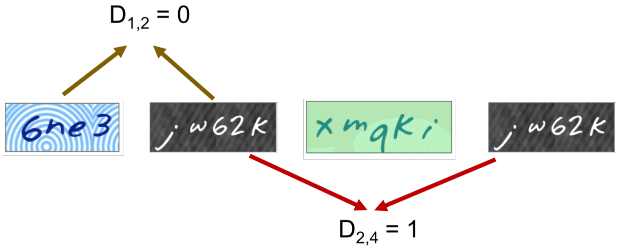

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$$\mathbb{E}[\mathbf{D}] = \sum_{i,j \in [m], i<j} \frac{1}{n} = \frac{\binom{m}{2}}{n} = \frac{m(m-1)}{2n}.$$

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You take $m = 1000$ samples. If the database size is as claimed ($n = 1,000,000$) then expected number of duplicates is:

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Concentration Inequalities: Bounds on the probability that a random variable deviates a certain distance from its mean.

- Useful in understanding how statistical tests perform, the behavior of randomized algorithms, the behavior of data drawn from different distributions, etc.

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The larger the deviation t, the smaller the probability.

BACK TO OUR APPLICATION

Expected number of duplicate CAPTCHAS:

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You see $\mathbf{D} = 10$ duplicates.

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This is pretty small and you feel pretty sure the number of unique CAPTCHAS is much less than 1000000.

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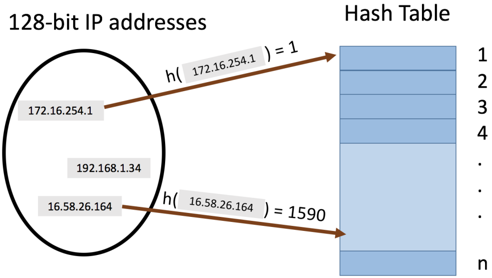
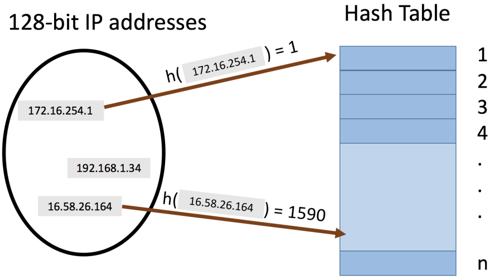
- *Static hashing* since we won't worry about insertion and deletion today.



- **hash function** $h : U \to [n]$ maps elements from the universe to indices $1, \cdots, n$ of an array.



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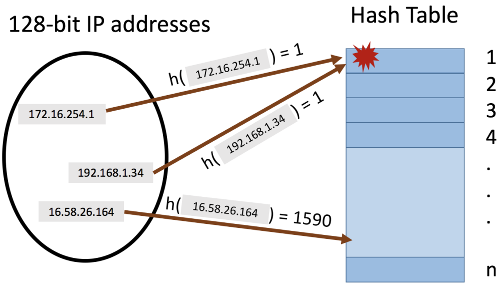
HASH TABLES



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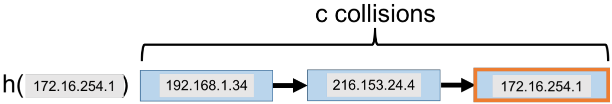
- Typically $|U| \gg n$. Many elements map to the same index.
- **Collisions:** when we insert m items into the hash table we may have to store multiple items in the same location (typically as a linked list).

Query runtime: $O(c)$ when the maximum number of collisions in a table entry is c (i.e., must traverse a linked list of size c).



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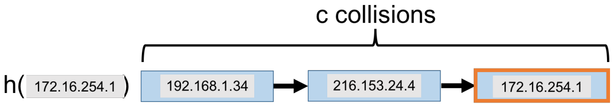

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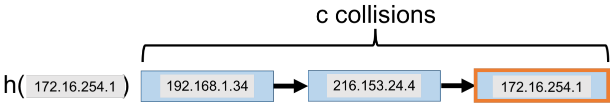

How Can We Bound c?

- In the worst case, could have $c = m$ (all items hash to the same location). In the best case, $c \approx m/n$.

RANDOM HASH FUNCTION

Let $\mathbf{h} : U \to [n]$ be a random hash function.

- Assume for the moment that $\mathbf{h}$ is fully independent, i.e., if $U = \{x_1, x_2, \ldots\}$ then
  a) $\Pr(\mathbf{h}(x_i) = j) = \frac{1}{n}$ for all $x_i \in U$ and $j \in [n]$ and
  b) all $\mathbf{h}(x_1), \mathbf{h}(x_2), \mathbf{h}(x_3) \ldots$ are all independent.

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- **Caveat 1:** It is *very expensive* to represent and compute fully independent random functions. Later, we will see how efficient hash functions can be used instead.
- **Caveat 2:** In practice, often suffices to use hash functions like MD5, SHA-2, etc. that 'look random enough'.

LINEARITY OF EXPECTATION

Let $\mathbf{C}_{i,j} = 1$ if items i and j collide ($\mathbf{h}(x_i) = \mathbf{h}(x_j)$), and 0 otherwise. The number of pairwise duplicates is:

$$\mathbf{C} = \sum_{i,j \in [m], i<j} \mathbf{C}_{i,j}.$$

x_i, x_j: pair of stored items, m: total number of stored items, n: hash table size, $\mathbf{C}$: total pairwise collisions in table, $\mathbf{h}$: random hash function.

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Identical to the CAPTCHA analysis!

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Pretty good but we are using $O(m^2)$ space to store m items.

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