COMPSCI 514: ALGORITHMS FOR DATA SCIENCE

Andrew McGregor

Lecture 20

Computing the SVD/eigendecomposition

- Efficient algorithms for SVD/eigendecomposition.
- High level: a glimpse into fast methods for linear algebraic computation, which are workhorses behind data science.

EFFICIENT EIGENDECOMPOSITION AND SVD

We have talked about the eigendecomposition and SVD as ways to compress data, to embed entities like words and documents, to compress/cluster non-linearly separable data.

How efficient are these techniques? Can they be run on massive datasets?

Power Method: The most fundamental iterative method for approximate SVD/eigendecomposition. Applies to computing $k = 1$ eigenvectors, but can be generalized to larger k.

Goal: Given symmetric $\mathbf{A} \in \mathbb{R}^{d \times d}$, with eigendecomposition $\mathbf{A} = \mathbf{V}\mathbf{\Lambda}\mathbf{V}^T$, find $\vec{z}$ which is an approximation to the top eigenvector $\vec{v}_1$ of $\mathbf{A}$.

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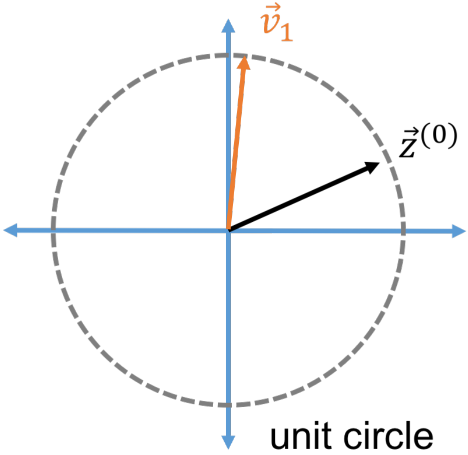
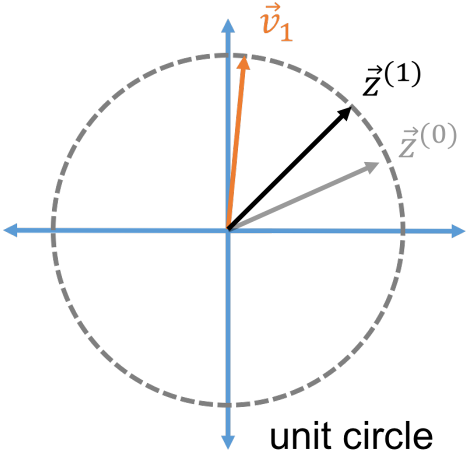
- **Initialize:** Choose $\vec{z}^{(0)}$ randomly. E.g. $\vec{z}^{(0)}(i) \sim \mathcal{N}(0, 1)$.
- For $i = 1, \ldots, t$
  - $\vec{z}^{(i)} := \mathbf{A} \cdot \vec{z}^{(i-1)}$
  - $\vec{z}_i := \frac{\vec{z}^{(i)}}{\|\vec{z}^{(i)}\|_2}$

  Return $\vec{z}_t$


$\vec{v}_1$
$\vec{z}^{(0)}$
unit circle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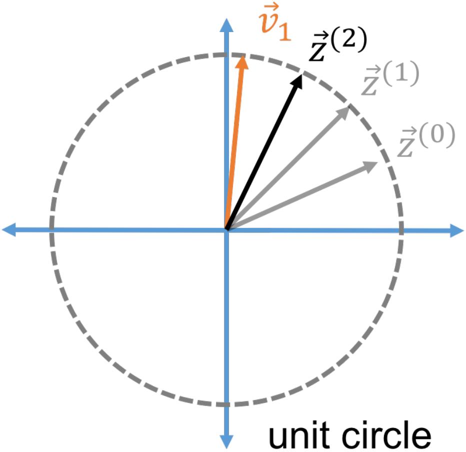
POWER METHOD



POWER METHOD ANALYSIS

Write $\vec{z}^{(0)}$ in $\mathbf{A}$'s eigenvector basis:

$$\vec{z}^{(0)} = c_1 \vec{v}_1 + c_2 \vec{v}_2 + \ldots + c_d \vec{v}_d.$$

$\mathbf{A} \in \mathbb{R}^{d \times d}$: input matrix with eigendecomposition $\mathbf{A} = \mathbf{V} \mathbf{\Lambda} \mathbf{V}^T$. $\vec{v}_1$: top eigenvector, being computed, $\vec{z}^{(i)}$: iterate at step i, converging to $\vec{v}_1$.

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Update step: $\vec{z}^{(i)} = \mathbf{A} \cdot \vec{z}^{(i-1)} = \mathbf{V\Lambda V}^T \cdot \vec{z}^{(i-1)}$ (then normalize)

$$\mathbf{V}^T\vec{z}^{(0)} =$$

$$\mathbf{\Lambda V}^T\vec{z}^{(0)} =$$

$$\vec{z}^{(1)} = \mathbf{V\Lambda V}^T \cdot \vec{z}^{(0)} =$$

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Claim 1 : Writing $\vec{z}^{(0)} = c_1\vec{v}_1 + c_2\vec{v}_2 + \ldots + c_d\vec{v}_d$,

$$\vec{z}^{(1)} = c_1 \cdot \lambda_1\vec{v}_1 + c_2 \cdot \lambda_2\vec{v}_2 + \ldots + c_d \cdot \lambda_d\vec{v}_d.$$

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$$\vec{z}^{(t)} = c_1 \cdot \lambda_1^t \vec{v}_1 + \mathbf{c}_2 \cdot \lambda_2^t \vec{v}_2 + \ldots + c_d \cdot \lambda_d^t \vec{v}_d.$$

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POWER METHOD CONVERGENCE

After t iterations, we have 'powered' up the eigenvalues, making the component in the direction of v_1 much larger, relative to the other components.

$$\vec{z}^{(0)} = c_1\vec{v}_1 + c_2\vec{v}_2 + \ldots + c_d\vec{v}_d \implies \vec{z}^{(t)} = c_1\lambda_1^t\vec{v}_1 + c_2\lambda_2^t\vec{v}_2 + \ldots + c_d\lambda_d^t\vec{v}_d$$

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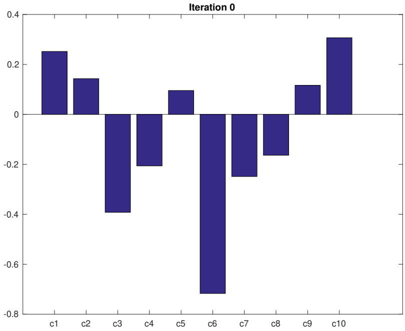

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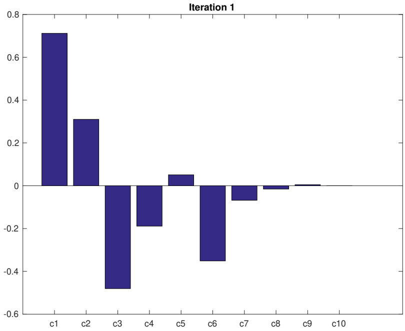

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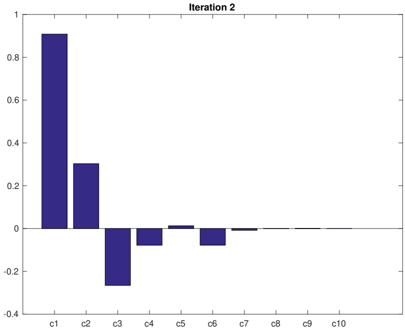

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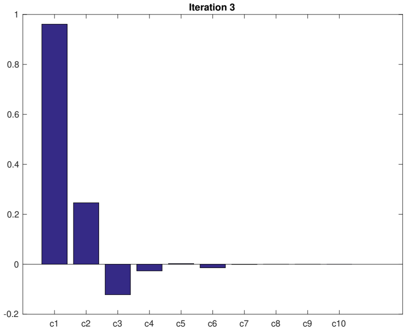

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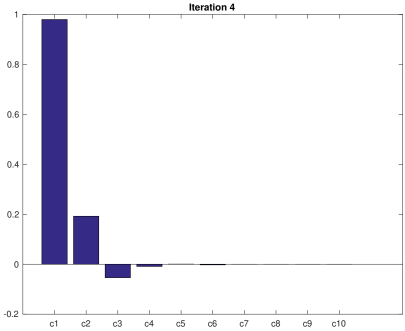

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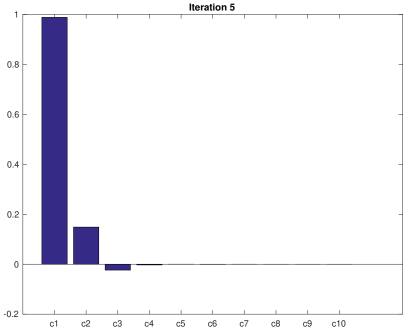

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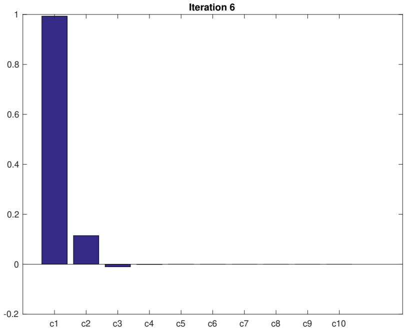

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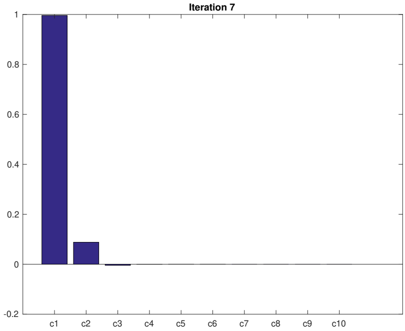

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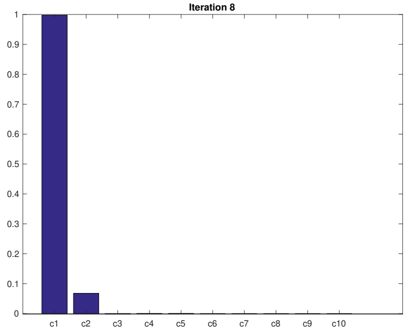

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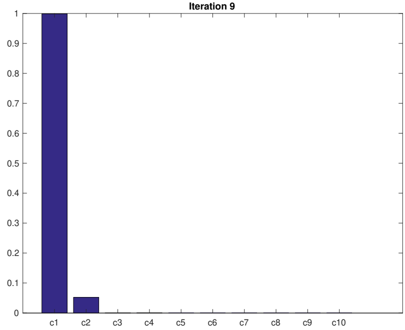

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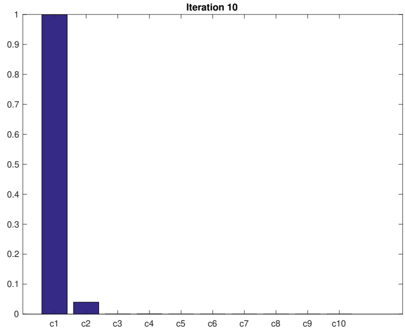

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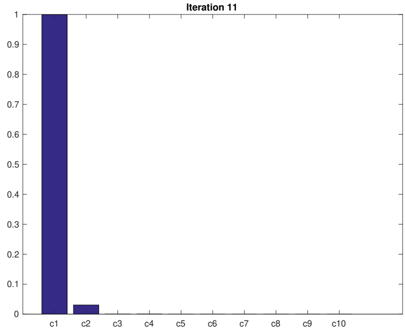

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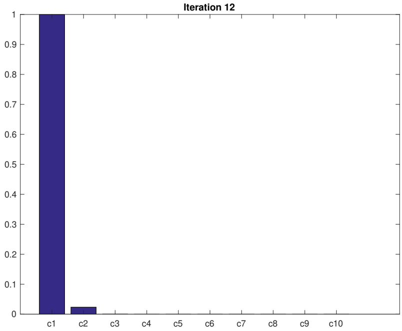

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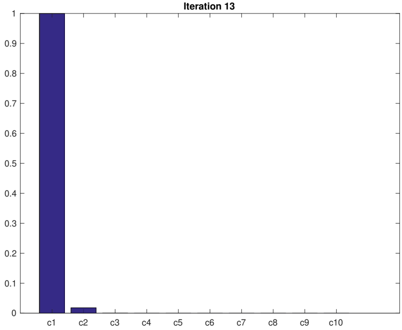

When will convergence be slow?

POWER METHOD SLOW CONVERGENCE

Slow Case: A has eigenvalues: $\lambda_1 = 1, \lambda_2 = .99, \lambda_3 = .9, \lambda_4 = .8, \ldots$

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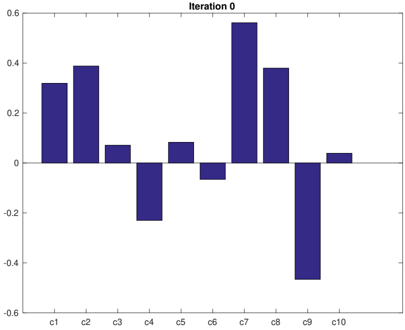

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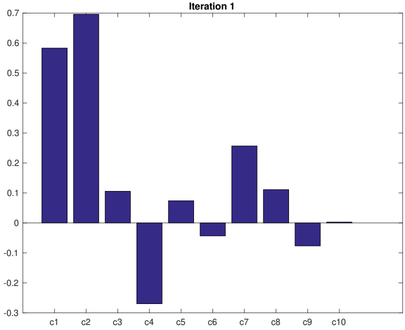

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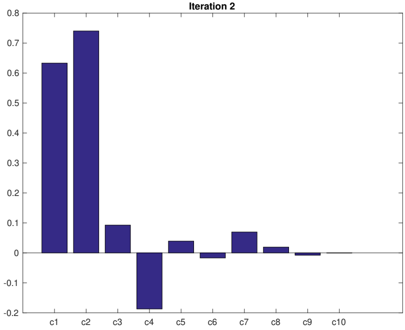

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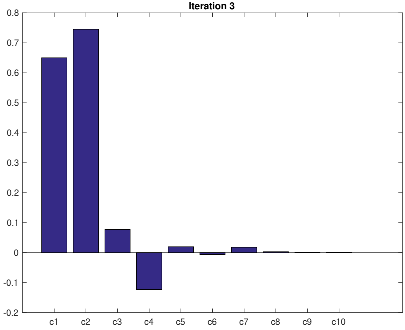

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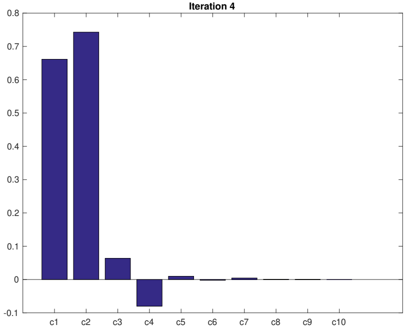

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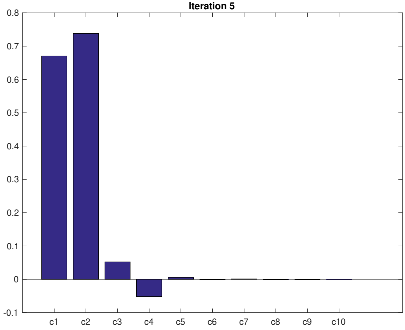

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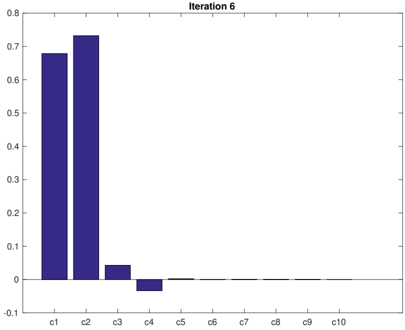

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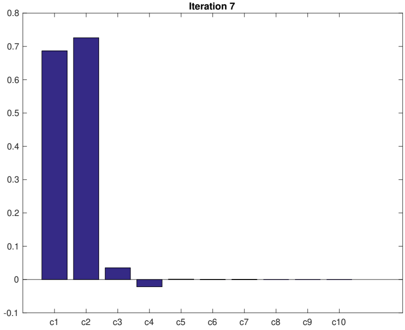

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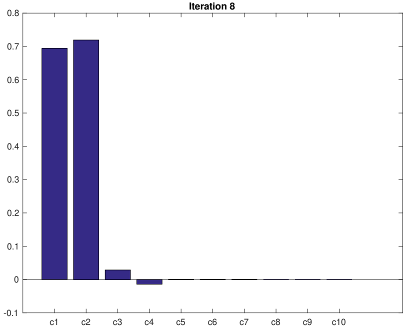

POWER METHOD SLOW CONVERGENCE

Slow Case: A has eigenvalues: $\lambda_1 = 1, \lambda_2 = .99, \lambda_3 = .9, \lambda_4 = .8, \ldots$

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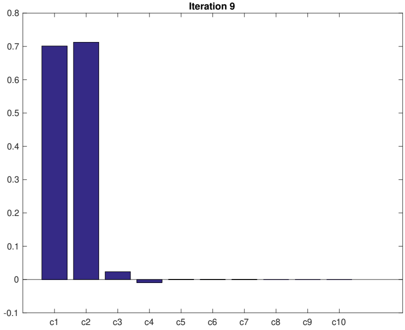

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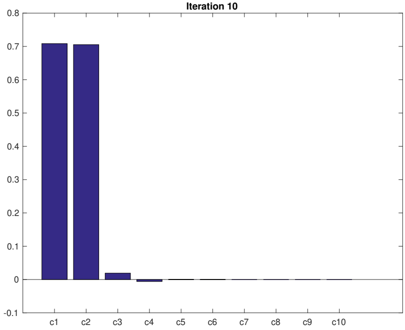

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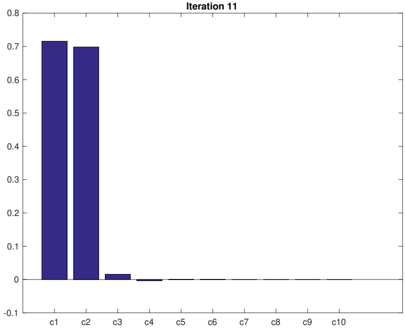

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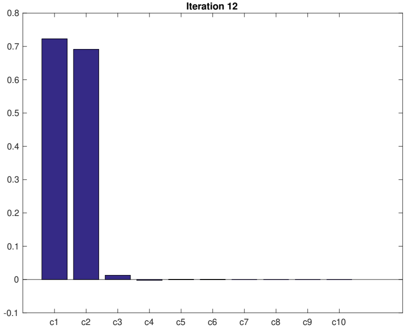

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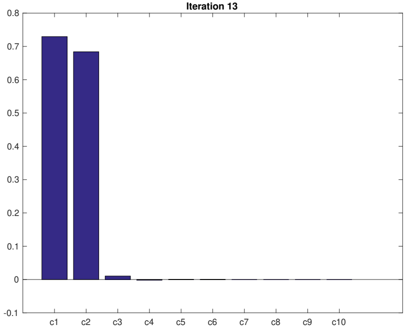

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How small must we set δ to ensure that $c_1\lambda_1^t$ dominates all other components and so $\vec{z}^{(t)}$ is very close to $\vec{v}_1$?

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Corollary:

$$\max_j \left| \frac{c_j}{c_1} \right| \leq O(d^2 \log d).$$

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- Claim: For $0 < c < \|x\|_2$:

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Setting $\delta = O\left(\frac{\epsilon}{d^3 \log d} \right)$ gives $\| \vec{z}^{(t)} - \vec{v}_1 \|_2 \leq \epsilon$.

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Theorem (Basic Power Method Convergence)

Let $\gamma = \frac{|\lambda_1| - |\lambda_2|}{|\lambda_1|}$ be the relative gap between the first and second eigenvalues. If Power Method is initialized with a random Gaussian vector $\vec{v}^{(0)}$ then, with high probability, after $t = O\left(\frac{\ln(d/\epsilon)}{\gamma}\right)$ steps:

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Total runtime: $O(t)$ matrix-vector multiplications. If $\mathbf{A} = \mathbf{X}^T\mathbf{X}$:

$$O\left(\text{nnz}(\mathbf{X}) \cdot \frac{\ln(d/\epsilon)}{\gamma} \cdot\right) = O\left(nd \cdot \frac{\ln(d/\epsilon)}{\gamma}\right).$$

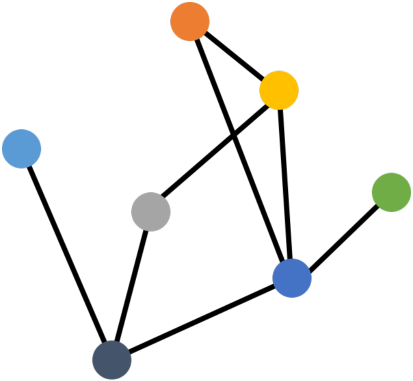
FINDING SECOND (ETC.) EIGENVECTOR

- If A has eigenvectors $v_1, \ldots, v_n$ with eigenvalues $\lambda_1, \ldots, \lambda_n$ $(|\lambda_1| \geq \ldots \geq |\lambda_n|)$ then

$$B = A - \lambda_1 v_1 v_1^T$$

  has eigenvectors $v_2, \ldots, v_n, v_1$ with eigenvectors $\lambda_2, \ldots, \lambda_n, 0$
- Hence, to find the second eigenvector of A, just apply the previous method to B.

Consider a random walk on a graph G with adjacency matrix $\mathbf{A}$.



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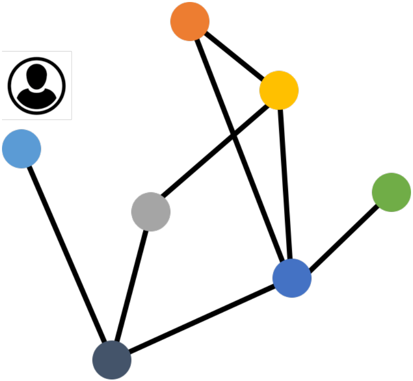

At each step, move to a random vertex, chosen uniformly at random from the neighbors of the current vertex.

CONNECTION TO RANDOM WALKS

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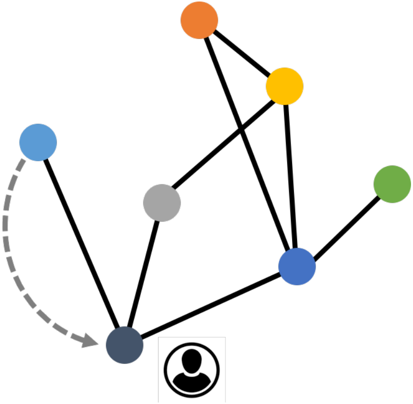

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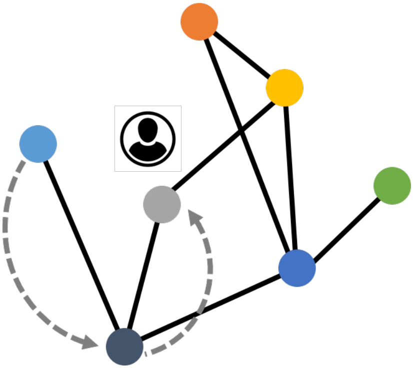

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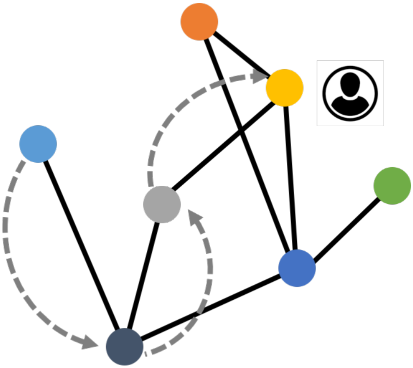

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- **Update:**

$$\Pr(\text{walk at i at step t}) = \sum_{j \in neigh(i)} \Pr(\text{walk at j at step t-1}) \cdot \frac{1}{degree(j)}$$

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- Like the power method, the time a random walk takes to converge to its stationary distribution (mixing time) is dependent on the gap between the top two eigenvalues of $\mathbf{D}^{-1/2}\mathbf{A}\mathbf{D}^{-1/2}$. The spectral gap.