# COMPSCI 514: ALGORITHMS FOR DATA SCIENCE

Andrew McGregor

Lecture 3

**Today:**

- Continue random hash functions and hash tables.
- See an application of random hashing to load balancing in distributed systems.
- Through this application learn about:
  - Chebyshev's inequality, which strengthens Markov's inequality.
  - The union bound, for understanding the probabilities of correlated random events.

# HASH TABLES

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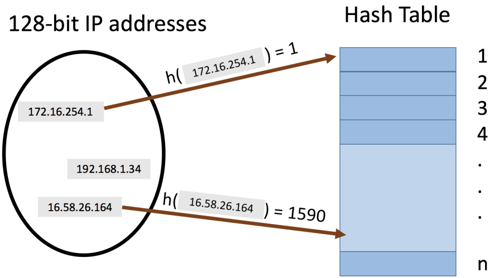
- *Static hashing* since we won't worry about insertion and deletion today.




- **hash function** $h : U \to [n]$ maps elements in universe $U = \{x_1, x_2, \ldots\}$ to indices of an array. Assume **h** is fully independent, i.e.,
  a) $\Pr(\mathbf{h}(x_i) = j) = \frac{1}{n}$ for all $x_i \in U$ and $j \in [n]$ and
  b) all $\mathbf{h}(x_1), \mathbf{h}(x_2), \mathbf{h}(x_3) \ldots$ are all independent.

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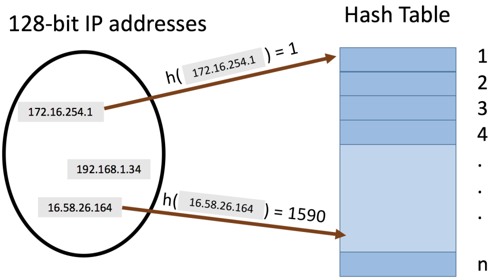


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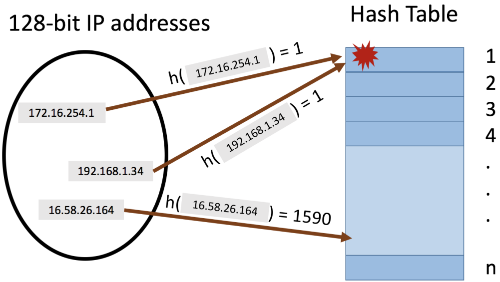


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Let $\mathbf{C}_{i,j} = 1$ if items $i$ and $j$ collide ($\mathbf{h}(x_i) = \mathbf{h}(x_j)$), and 0 otherwise. The number of pairwise collisions is:

$$\mathbf{C} = \sum_{i,j \in [m], i<j} \mathbf{C}_{i,j}.$$

$x_i, x_j$: pair of stored items, $m$: total number of stored items, $n$: hash table size, $\mathbf{C}$: total pairwise collisions in table, $\mathbf{h}$: random hash function.

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Identical to the CAPTCHA analysis!

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Pretty good but we are using $O(m^2)$ space to store $m$ items.

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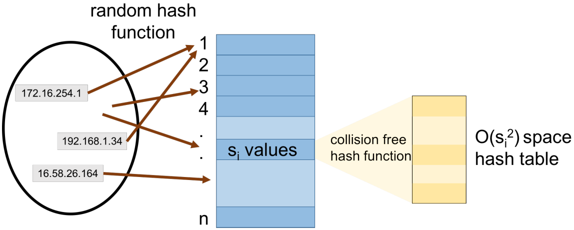

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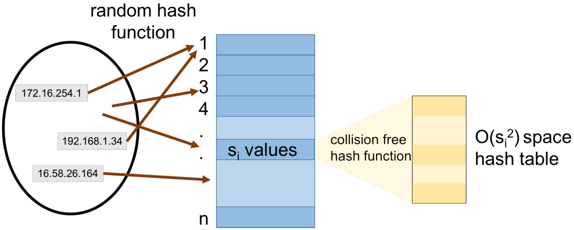


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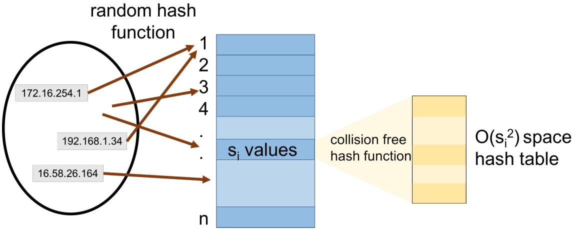


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- **Previously:** Showed that a random function is collision free with probability $\geq \frac{7}{8}$ so can just generate a random hash function and check if it is collision free.

# SPACE USAGE

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**Collisions again!**

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**Efficient Alternative:** Let $p$ be a prime with $p \geq |U|$. Choose random $\mathbf{a}, \mathbf{b} \in [p]$ with $\mathbf{a} \neq 0$. Let:

$$\mathbf{h}(x) = (\mathbf{a}x + \mathbf{b} \mod p) \mod n.$$

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**Remember:** A fully random hash function is both 2-universal and pairwise independent. But it is not efficiently implementable.

# NEXT STEP

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2. Then we'll show how a simple twist on Markov's can give a much stronger result.

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(by plugging in the random variable $\mathbf{X} - \mathbb{E}[\mathbf{X}]$)