COMPSCI 514: ALGORITHMS FOR DATA SCIENCE

Andrew McGregor

Lecture 4

LAST TIME

Last Class:

- 2-Level Hashing Analysis (linearity of expectation and Markov's inequality)
- 2-universal and pairwise independent hash functions
- Chebyshev: $\Pr(|\mathbf{X} - \mathbb{E}[\mathbf{X}]| \geq t) \leq \text{Var}[\mathbf{X}]/t^2$

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This Time:

- Random hashing for load balancing. Motivating:
  - Stronger concentration inequalities: Chebyshev's inequality, exponential tail bounds, and their connections to the law of large numbers.
  - The union bound.

- Suppose random variable X and can be written as

$$X = A_1 + A_2 + \ldots + A_n$$

where each A_i are independent indicator variables with $\Pr(A_i) = p$.

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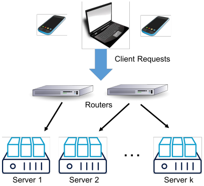
- Note $\mathbb{E}[A_i] = p$ and $\mathrm{Var}[A_i] = \mathbb{E}[A_i^2] - \mathbb{E}[A_i]^2 = p - p^2$.
- By linearity of expectation and variance,

$$\mathbb{E}[X] = np \qquad \mathrm{Var}[X] = np(1-p) \ .$$

Randomized Load Balancing:



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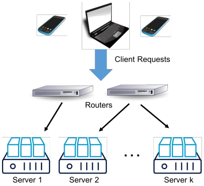

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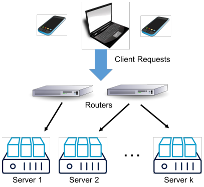

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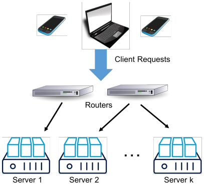

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- Variance:

$$\text{Var}[\mathbf{R}_i] = \text{Var}[\sum_{j=1}^{n} \mathbb{I}_{\text{request } j \text{ assigned to } i}] = \sum_{j=1}^{n} \text{Var}[\mathbb{I}_{j \text{ assigned to } i}] = n\left(\frac{1}{k} - \frac{1}{k^2}\right)$$

What is the probability that the maximum server load exceeds $2 \cdot \mathbb{E}[\mathbf{R}_i] = \frac{2n}{k}$. I.e., some server is overloaded if each has $\frac{2n}{k}$ capacity?

MAXIMUM SERVER LOAD

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We want to upper bound:

$$\Pr\left(\max_i(\mathbf{R}_i) \geq \frac{2n}{k}\right) = \Pr\left(\left[\mathbf{R}_1 \geq \frac{2n}{k}\right] \text{ or } \dots \text{ or } \left[\mathbf{R}_k \geq \frac{2n}{k}\right]\right)$$
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How do we do this since $\mathbf{R}_1, \ldots, \mathbf{R}_k$ are not independent?

Union Bound: For any random events $A_1, A_2, ..., A_k$,

$$\Pr(A_1 \cup A_2 \cup \ldots \cup A_k) \leq \Pr(A_1) + \Pr(A_2) + \ldots + Pr(A_k).$$

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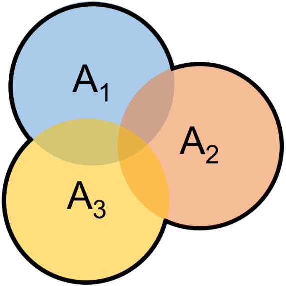

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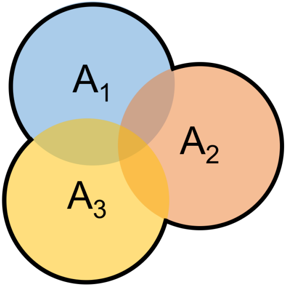

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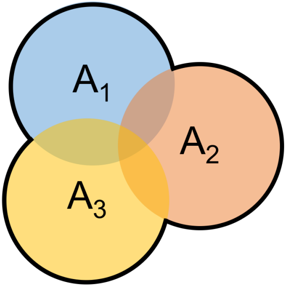

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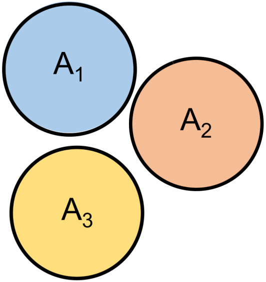

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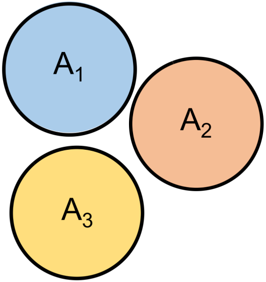

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On the first problem set, you will prove the union bound, as a consequence of Markov's inquality.

APPLYING THE UNION BOUND

What is the probability that the maximum server load exceeds $2 \cdot \mathbb{E}[\mathbf{R}_i] = \frac{2n}{k}$.
I.e., that some server is overloaded if we give each $\frac{2n}{k}$ capacity?

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n: total number of requests, k: number of servers randomly assigned requests, $\mathbf{R}_i$: number of requests assigned to server i. $\mathbb{E}[\mathbf{R}_i] = \frac{n}{k}$. $\text{Var}[\mathbf{R}_i] = \frac{n}{k}$.

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\end{aligned}$$

As long as $k \ll \sqrt{n}$, the maximum server load will be small (compared to the expected load) with good probability.

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BACK TO CHEBYSHEV'S INEQUALITY

$$\Pr(|\mathbf{X} - \mathbb{E}[\mathbf{X}]| \geq t) \leq \frac{\mathrm{Var}[\mathbf{X}]}{t^2}$$

$\mathbf{X}$: any random variable, t, s: any fixed numbers.

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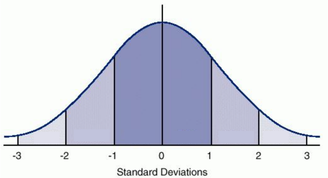
What is the probability that **X** falls s standard deviations from it's mean?



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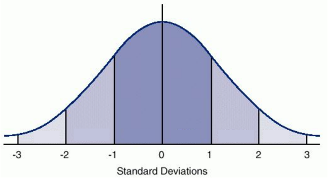

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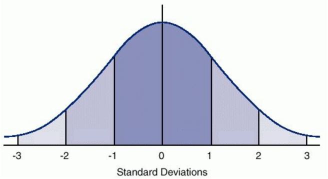

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Why is this so powerful?

$\mathbf{X}$: any random variable, t, s: any fixed numbers.

Consider drawing independent identically distributed (i.i.d.) random variables $\mathbf{X}_1, \ldots, \mathbf{X}_n$ with mean μ and variance σ^2.

LAW OF LARGE NUMBERS

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- Cannot show from vanilla Markov's inequality.

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SERVER LOAD AND LAW OF LARGE NUMBERS

The number of servers must be small compared to the number of requests ($k = O(\sqrt{n})$) for the maximum load to be bounded in comparison to the expected load with good probability.

- There are many requests routed to a relatively small number of servers so the load seen on each server is close to what is expected via law of large numbers.

n: total number of requests, k: number of servers randomly assigned requests.

Questions on union bound, Chebyshev's inequality, random hashing?

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$$\mathbb{E}[\mathbf{H}] = \frac{n}{2} = 50 \text{ and } \mathrm{Var}[\mathbf{H}] =$$

FLIPPING COINS

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$\Pr(\mathbf{H} \geq 60) \leq .833$	$\Pr(\mathbf{H} \geq 60) \leq .25$	$\Pr(\mathbf{H} \geq 60) = 0.0284$
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$\mathbf{H}$ has a simple Binomial distribution, so can compute these probabilities exactly.

TIGHTER CONCENTRATION BOUNDS

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- What if we just apply Markov's inequality to even higher moments?

A FOURTH MOMENT BOUND

Consider any random variable $\mathbf{X}$:

$$\Pr(|\mathbf{X} - \mathbb{E}[\mathbf{X}]| \geq t) = \Pr\left((\mathbf{X} - \mathbb{E}[\mathbf{X}])^4 \geq t^4\right)$$

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$$\mathbb{E}\left[(\mathbf{H} - \mathbb{E}[\mathbf{H}])^4\right] = \mathbb{E}\left[\left(\sum_{i=1}^{100} \mathbf{H}_i - 50\right)^4\right]$$

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- Apply Fourth Moment Bound: $\Pr(|\mathbf{H} - \mathbb{E}[\mathbf{H}]| \geq t) \leq \frac{1862.5}{t^4}$.

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- We aren't restricted to applying Markov's to $|\mathbf{X} - \mathbb{E}[\mathbf{X}]|^k$ for some k. Can apply to any monotonic function $f(|\mathbf{X} - \mathbb{E}[\mathbf{X}]|)$.
- Why monotonic? $\Pr(|\mathbf{X} - \mathbb{E}[\mathbf{X}]| > t) = \Pr(f(|\mathbf{X} - \mathbb{E}[\mathbf{X}]|) > f(t))$.

- **Moment Generating Function:** Consider for any $r > 0$:

$$M_r(\mathbf{X}) = e^{r \cdot (\mathbf{X} - \mathbb{E}[\mathbf{X}])} = \sum_{k=0}^{\infty} \frac{r^k (\mathbf{X} - \mathbb{E}[\mathbf{X}])^k}{k!}$$

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- Weighted sum of all moments (r controls the weights) and choosing r appropriately lets one prove a number of very powerful exponential concentration bounds such as Chernoff, Bernstein, Hoeffding, Azuma, Berry-Esseen, etc.

BERNSTEIN INEQUALITY

Bernstein Inequality: Consider independent random variables $\mathbf{X}_1, \ldots, \mathbf{X}_n \in [-M, M]$. Let $\mu = \mathbb{E}[\sum_{i=1}^n \mathbf{X}_i]$ and $\sigma^2 = \text{Var}[\sum_{i=1}^n \mathbf{X}_i]$. For any $t \geq 0$:

$$\Pr\left(\left|\sum_{i=1}^n \mathbf{X}_i - \mu\right| \geq t\right) \leq 2\exp\left(-\frac{t^2}{2\sigma^2 + \frac{4}{3}Mt}\right).$$

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- An exponentially stronger dependence on s!

COMPARISION TO CHEBYSHEV

Consider again bounding the number of heads **H** in $n = 100$ independent coin flips.

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Getting much closer to the true probability.

H: total number heads in 100 random coin flips. $\mathbb{E}[\mathbf{H}] = 50$.

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A useful variation for binary (indicator) random variables is:

Chernoff Bound (simplified version): Consider independent random variables $\mathbf{X}_1, \ldots, \mathbf{X}_n$ taking values in $\{0, 1\}$. Let $\mu = \mathbb{E}[\sum_{i=1}^n \mathbf{X}_i]$. For any $\delta \geq 0$

$$\Pr\left(\left|\sum_{i=1}^n \mathbf{X}_i - \mu\right| \geq \delta\mu\right) \leq 2\exp\left(-\frac{\delta^2\mu}{2+\delta}\right).$$