COMPSCI 514: ALGORITHMS FOR DATA SCIENCE

Andrew McGregor

Lecture 5

A variation of the Bernstein inequality for binary random variables is:

Chernoff Bound (simplified version): Consider independent random variables $\mathbf{X}_1, \ldots, \mathbf{X}_n$ taking values in $\{0, 1\}$. Let $\mu = \mathbb{E}[\sum_{i=1}^n \mathbf{X}_i]$. For any $\delta \geq 0$

$$\Pr\left(\left|\sum_{i=1}^n \mathbf{X}_i - \mu\right| \geq \delta\mu\right) \leq 2\exp\left(-\frac{\delta^2 \mu}{2+\delta}\right).$$

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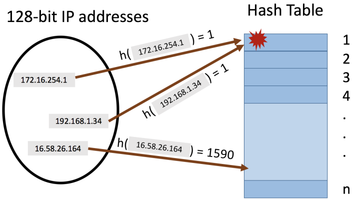
$$\Pr\left(\left|\sum_{i=1}^{n} \mathbf{X}_i - \mu\right| \geq \delta\mu\right) \leq 2\exp\left(-\frac{\delta^2\mu}{2+\delta}\right).$$

As δ gets larger and larger, the bound falls of exponentially fast.

RETURN TO RANDOM HASHING



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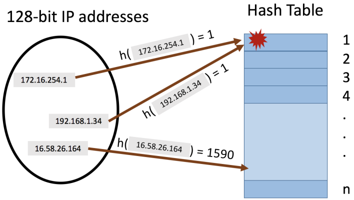

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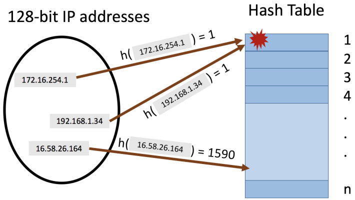

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What will be the maximum number of items hashed into the same location?

MAXIMUM LOAD IN RANDOMIZED HASHING

Let $\mathbf{S}_i$ be the number of items hashed into position i and $\mathbf{S}_{i,j}$ be 1 if x_j is hashed into bucket i ($\mathbf{h}(x_j) = i$) and 0 otherwise.

m: total number of items hashed and size of hash table. $x_1, \ldots, x_m$: the items. $\mathbf{h}$: random hash function mapping $x_1, \ldots, x_m \to [m]$.

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$$\mathbb{E}[\mathbf{S}_i] = \sum_{j=1}^{m} \mathbb{E}[\mathbf{S}_{i,j}] = m \cdot \frac{1}{m} = 1$$

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By the Chernoff Bound: for any $\delta \geq 0$,

$$\Pr(\mathbf{S}_i \geq 1 + \delta) \leq \Pr\left(\left|\sum_{i=1}^{n} \mathbf{S}_{i,j} - 1\right| \geq \delta \cdot \mu\right) \leq 2\exp\left(-\frac{\delta^2}{2+\delta}\right)$$

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- Using Chebyshev's inequality could only show the maximum load is bounded by $O(\sqrt{m})$ with good probability (good exercise).
- The Chebyshev bound holds even with a pairwise independent hash function. The stronger Chernoff-based bound can be shown to hold with a *k-wise independent hash function* for $k = O(\log m)$.

Questions on Exponential Concentration Bounds?

This concludes the probability foundations part of the course.
On to algorithms...

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Solution: Bloom filters (repeated random hashing). Will use much less space than a hash table.

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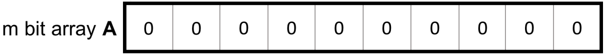

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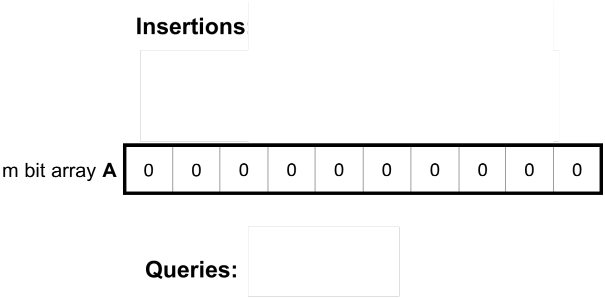

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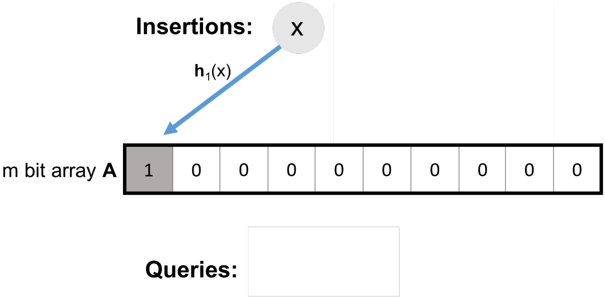

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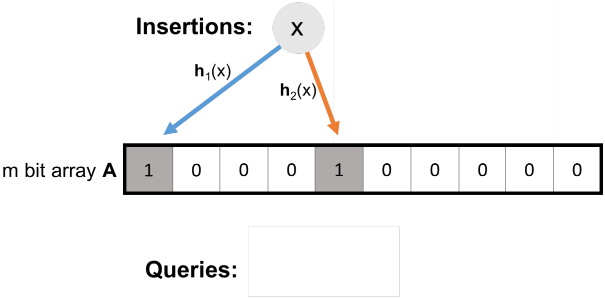

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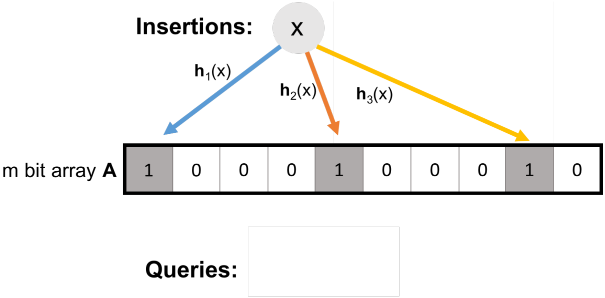

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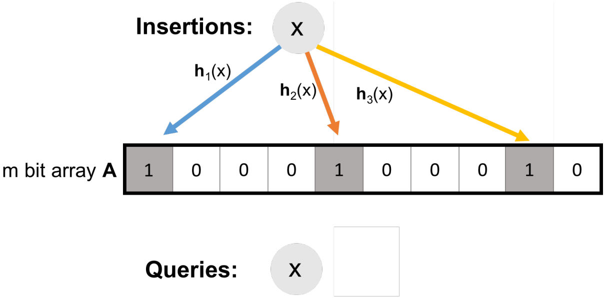

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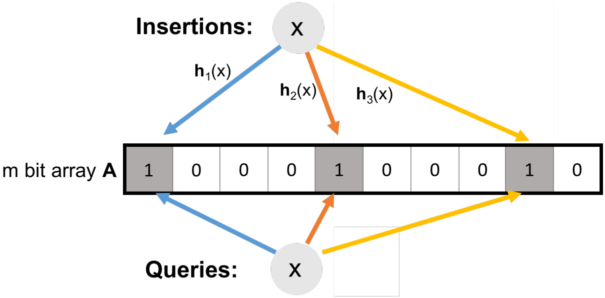

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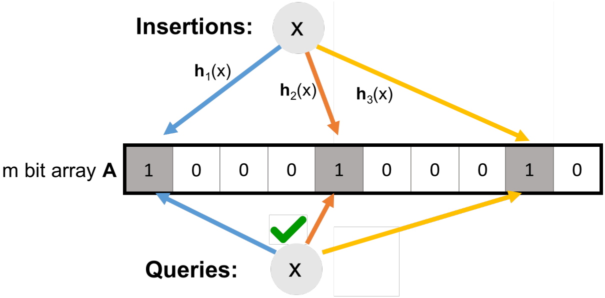

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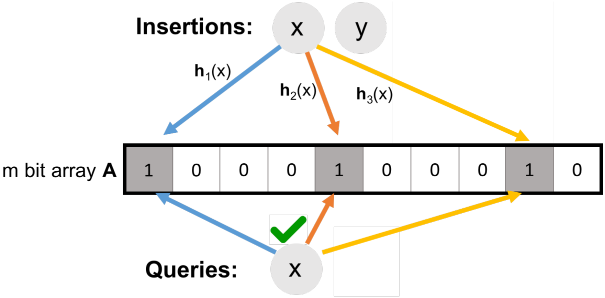

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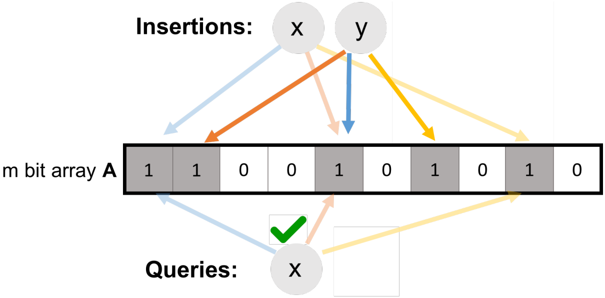

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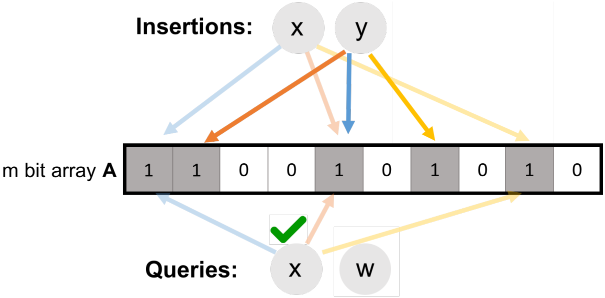

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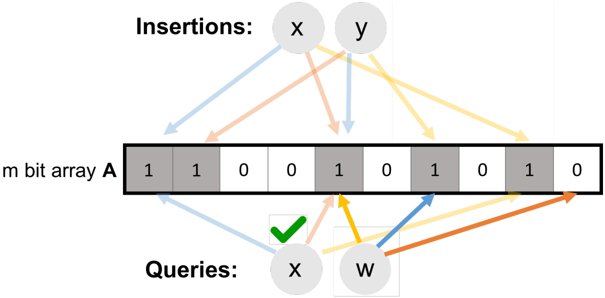

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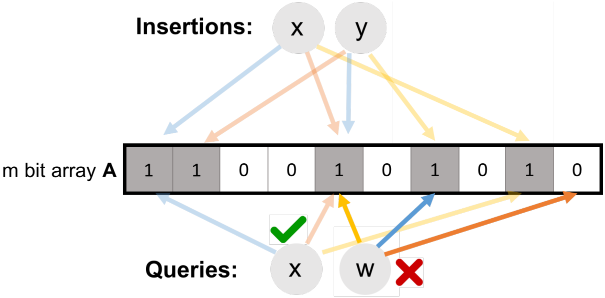

No false negatives. False positives more likely with more insertions.

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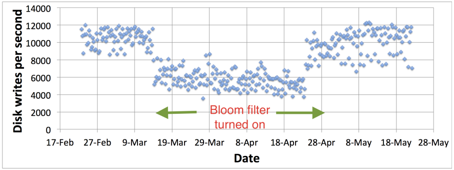

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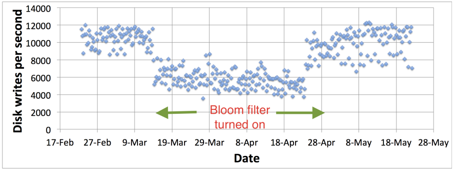

- A Bloom Filter can be used to approximately track the url's you've seen before without have to store them all!

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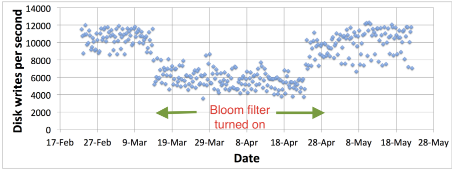

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APPLICATIONS: DETERMINING WHEN TO CACHE

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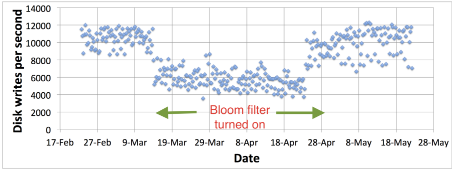

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- **False positive:** A new url (possible one-hit-wonder) is cached. If the bloom filter has a false positive rate of $\delta = .05$, the number of cached one-hit-wonders will be reduced by at least 95%.

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> n: total number items in filter, m: number of bits in filter, k: number of random hash functions, $\mathbf{h}_1, \ldots \mathbf{h}_k$: hash functions, A: bit array, δ: false positive rate.

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Let T be the number of zeros in the array after n inserts. Then,

$$E[T] = m\left(1 - \frac{1}{m}\right)^{kn} \approx me^{-\frac{kn}{m}}$$

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CORRECT ANALYSIS SKETCH

If T is the number of 0 entries, for a non-inserted element w:

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&\Pr(A[\mathbf{h}_1(w)] = \ldots = A[\mathbf{h}_k(w)] = 1) \\
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- How small is T/m? Note that $\frac{T}{m} \geq \frac{m-nk}{m} \approx e^{-\frac{kn}{m}}$ when $kn \ll m$. More generally, it can be shown that $T/m = \Omega\left(e^{-\frac{kn}{m}}\right)$ via Theorem 2 of:

  `cglab.ca/~morin/publications/ds/bloom-submitted.pdf`

FALSE POSITIVE RATE

False Positive Rate: with m bits of storage, k hash functions, and n items inserted $\delta \approx \left(1 - e^{\frac{-kn}{m}}\right)^k$.

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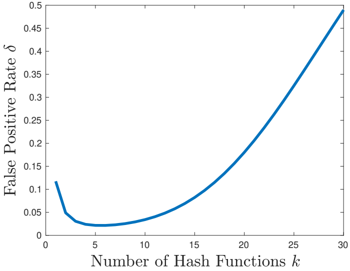

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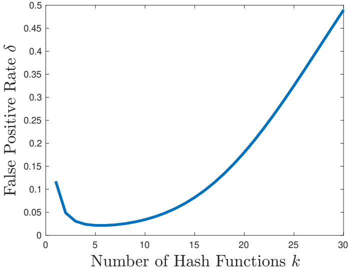

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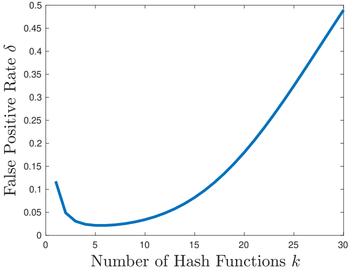

- Can differentiate to show optimal number of hashes is $k = \ln 2 \cdot \frac{m}{n}$ (rounded to the nearest integer). This gives $\delta \approx 1/2^{(m/n)\ln 2}$.
- Balances between filling up the array with too many hashes and having enough hashes so that even when the array is pretty full, a new item is unlikely to have all its bits set (yield a false positive)