COMPSCI 514: ALGORITHMS FOR DATA SCIENCE

Andrew McGregor

Lecture 6

Stream Processing: Have a massive dataset X with n items $x_1, x_2, \ldots, x_n$ that arrive in a continuous stream. Not nearly enough space to store all the items (in a single location).

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- Compared to traditional algorithm design, which focuses on minimizing runtime, the big question here is how much space is needed to answer queries of interest.

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Google Sawzall, Facebook Presto, Apache Drill, Twitter Algebird

DISTINCT ELEMENTS IDEAS

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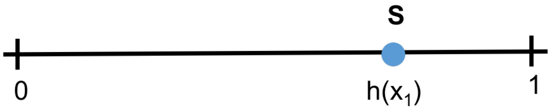

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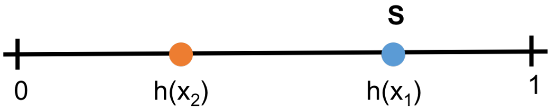

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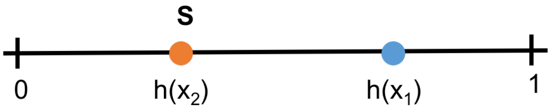

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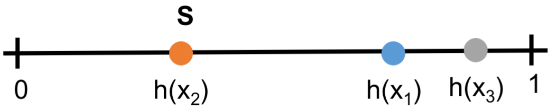

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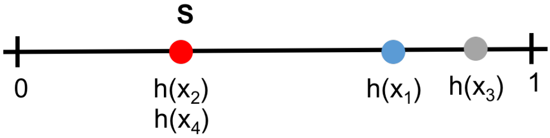

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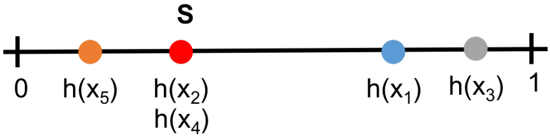

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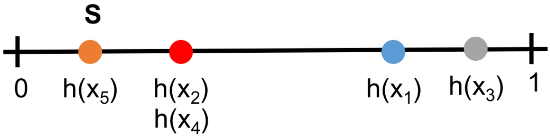

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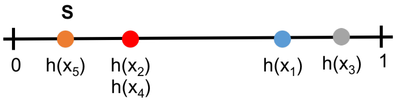

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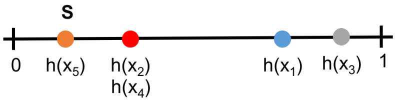

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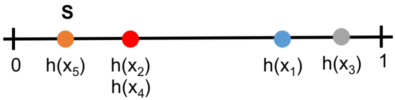

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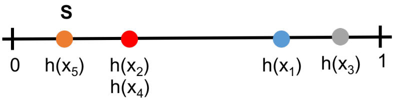

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- Same idea as Flajolet-Martin algorithm and HyperLogLog, except they use discrete hash functions.

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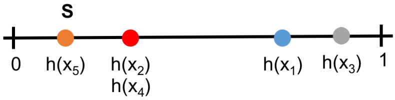

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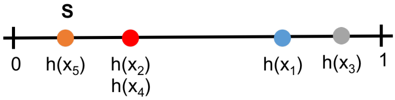

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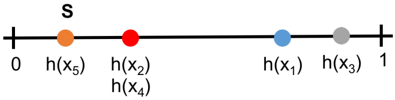

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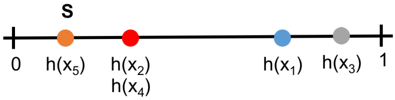

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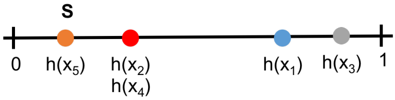

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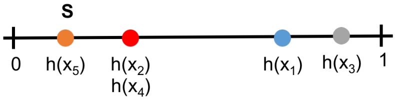

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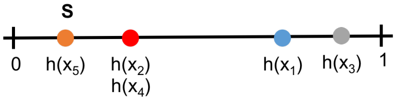

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- **Exercise:** Approximation is robust, i.e., if $|\mathbf{s} - \mathbb{E}[\mathbf{s}]| \leq \epsilon \cdot \mathbb{E}[\mathbf{s}]$ for any $\epsilon \in (0, 1/2)$,

$$(1-4\epsilon)d \leq \widehat{\mathbf{d}} \leq (1+4\epsilon)d$$

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Leverage the law of large numbers: improve accuracy via repeated independent trials.

Hashing for Distinct Elements (Improved):

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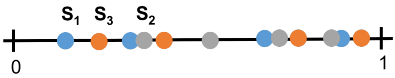

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ANALYSIS

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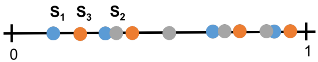

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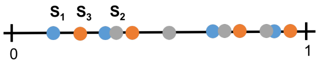

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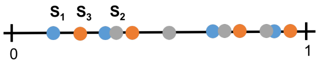

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- $\delta = 5\%$ failure rate gives a factor 20 overhead in space complexity.

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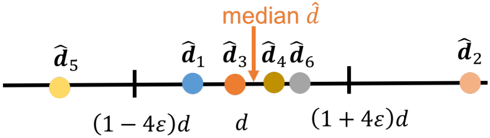

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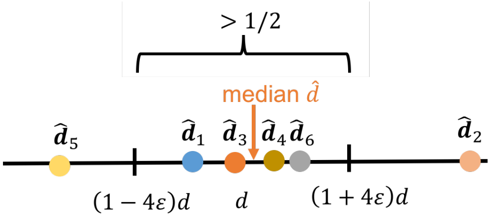

- If $> 1/2$ of trials fall in $[(1-4\epsilon)d, (1+4\epsilon)d]$, then the median will.

THE MEDIAN TRICK

- $\widehat{\mathbf{d_1}}, \ldots, \widehat{\mathbf{d_t}}$ are the outcomes of the t trials, each falling in

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$$\Pr\left(|\mathbf{X} - \mathbb{E}[\mathbf{X}]| \geq \frac{1}{3}\mathbb{E}[\mathbf{X}]\right) \leq 2\exp\left(-\frac{\frac{1}{3}^2 \cdot \frac{3}{4}t}{2+1/3}\right) = e^{-\Theta(t)}.$$

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with probability at least $3/4$. Let $\widehat{\mathbf{d}} = median(\widehat{\mathbf{d}}_1, \ldots, \widehat{\mathbf{d}}_t)$.

What is the probability that the median $\widehat{\mathbf{d}}$ falls in $[(1-4\epsilon)d, (1+4\epsilon)d]$?

- Let $\mathbf{X}$ be the # of trials falling in $[(1-4\epsilon)d, (1+4\epsilon)d]$. $\mathbb{E}[\mathbf{X}] \geq \frac{3}{4} \cdot t$.

$$\Pr\left(\widehat{\mathbf{d}} \notin [(1-4\epsilon)d, (1+4\epsilon)d]\right) \leq \Pr\left(\mathbf{X} \leq \frac{1}{2} \cdot t\right) \leq \Pr\left(\mathbf{X} \leq \frac{2}{3} \cdot \mathbb{E}[\mathbf{X}]\right)$$

Apply Chernoff bound:

$$\Pr\left(|\mathbf{X} - \mathbb{E}[\mathbf{X}]| \geq \frac{1}{3}\mathbb{E}[\mathbf{X}]\right) \leq 2\exp\left(-\frac{\frac{1}{3}^2 \cdot \frac{3}{4}t}{2+1/3}\right) = e^{-\Theta(t)}.$$

- Setting $t = O(\log(1/\delta))$ gives failure probability $e^{-\log(1/\delta)} = \delta$.

Upshot: The median of $t = O(\log(1/\delta))$ independent runs of the hashing algorithm for distinct elements returns

$$\widehat{\mathbf{d}} \in [(1-4\epsilon)d, (1+4\epsilon)d]$$

with probability at least $1-\delta$.

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A note on the median: The median is often used as a robust alternative to the mean, when there are outliers (e.g., heavy tailed distributions, corrupted data).