COMPSCI 514: ALGORITHMS FOR DATA SCIENCE

Andrew McGregor

Lecture 9

THE FREQUENT ITEMS DATA STREAM PROBLEM

k-**Frequent Items (Heavy-Hitters) Problem**: Consider a stream of n items $x_1, \ldots, x_n$ (with possible duplicates). Return any item at appears at least $\frac{n}{k}$ times.

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- Trivial with $O(n)$ space: Store the count for each item and return the one that appears $\geq n/k$ times.
- Can we do it with less space? I.e., without storing all n items?

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Generally want very fast detection, without having to scan through database/logs. That is we want to maintain a running list of frequent items that appear in a stream.

APPROXIMATE FREQUENT ELEMENTS

Issue: No algorithm using $o(n)$ space can output just the items with frequency $\geq n/k$. Hard to tell between an item with frequency n/k (should be output) and $n/k - 1$ (should not be output).

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- An example of relaxing to a ‘promise problem’: for items with frequencies in $[(1-\epsilon) \cdot \frac{n}{k}, \frac{n}{k}]$ no output guarantee.

FREQUENT ELEMENTS WITH COUNT-MIN SKETCH

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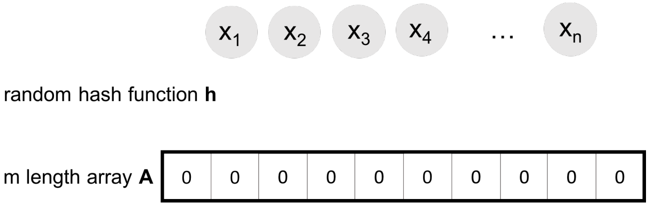

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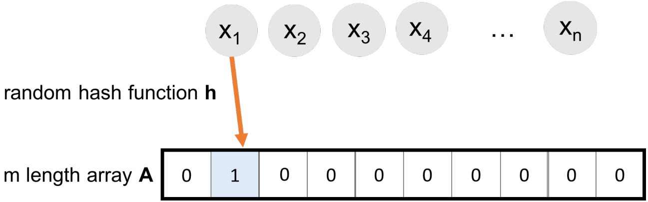

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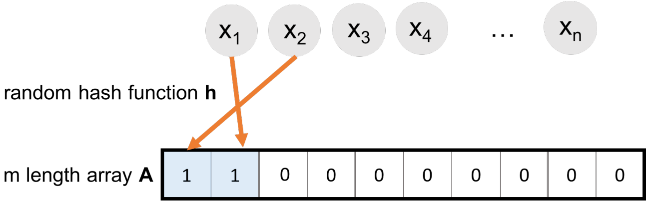

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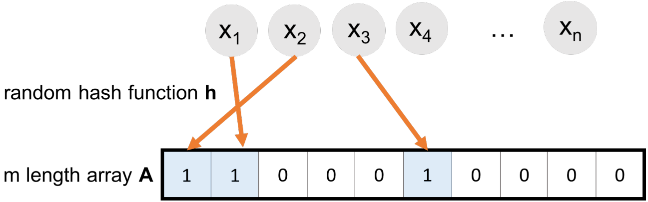

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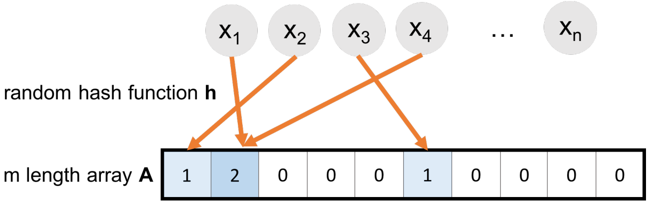

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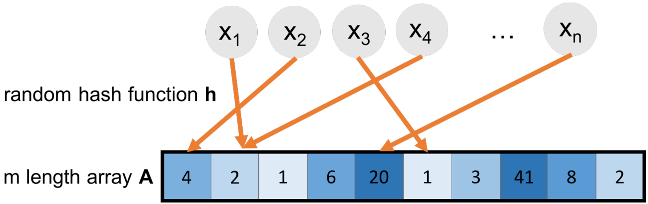

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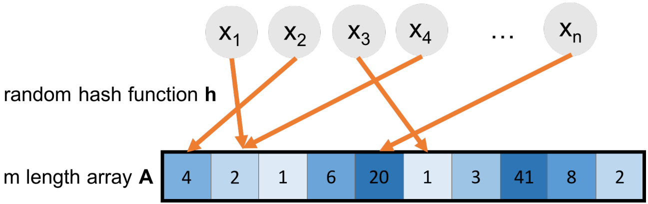
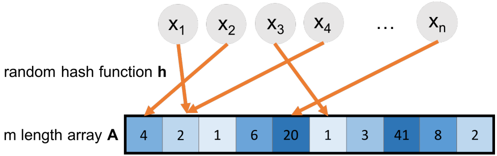


Will use $A[\mathbf{h}(x)]$ to estimate $f(x) = |\{i : x_i = x\}|$, the frequency of x in the stream.

COUNT-MIN SKETCH ACCURACY



Use $A[\mathbf{h}(x)]$ to estimate $f(x)$.

Claim 1: We always have $A[\mathbf{h}(x)] \geq f(x)$. Why?

$f(x)$: frequency of x in the stream (i.e., number of items equal to x). **h**: random hash function. m: size of Count-min sketch array.

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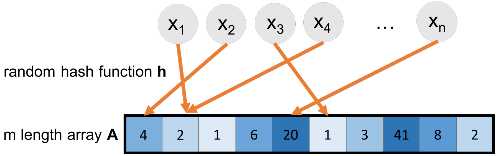

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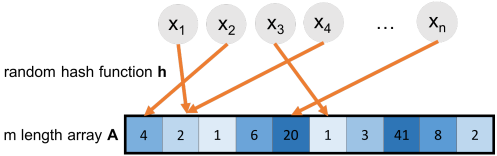

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- $A[\mathbf{h}(x)]$ counts the number of occurrences of any y with $\mathbf{h}(y) = \mathbf{h}(x)$, including x itself.
- $A[\mathbf{h}(x)] = f(x) + \sum_{y \neq x : \mathbf{h}(y) = \mathbf{h}(x)} f(y)$.

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COUNT-MIN SKETCH ACCURACY

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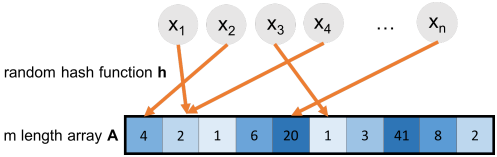

Claim: For any x, with probability at least $1/2$,

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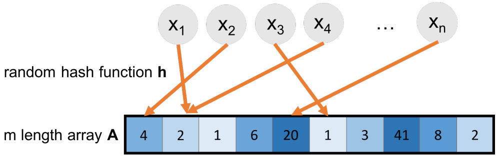

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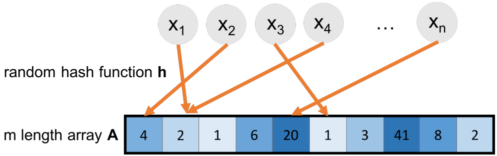

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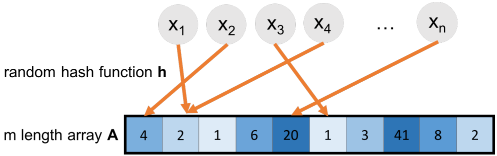

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How can we improve the success probability? Repetition.

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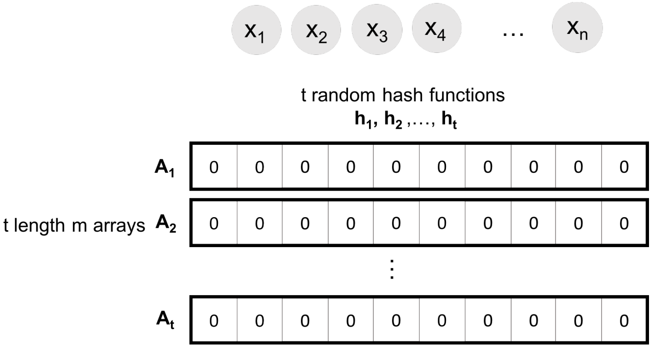
x_1 x_2 x_3 x_4 … x_n
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t length m arrays
$\mathbf{A_1}$
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0 0 0 0 0 0 0 0 0 0
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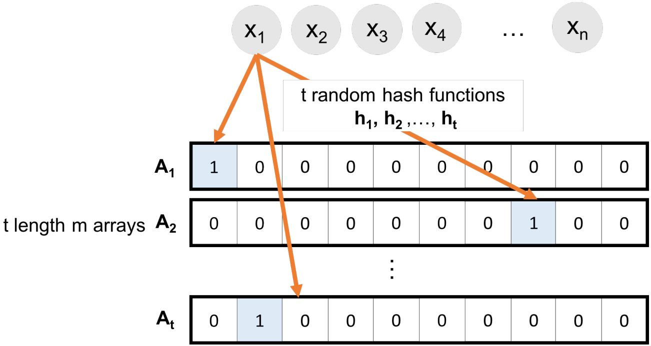
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⋮
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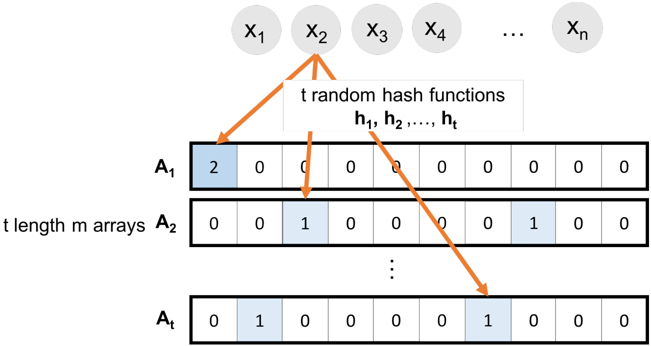
COUNT-MIN SKETCH ACCURACY



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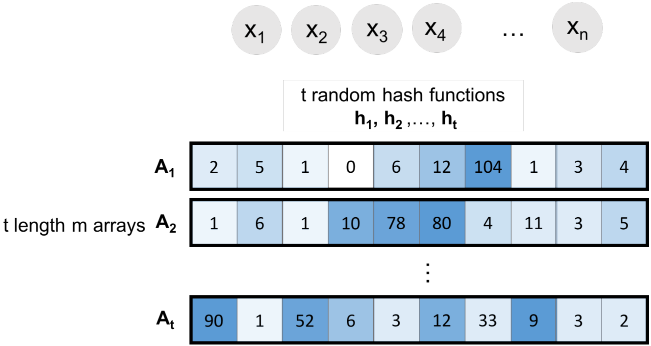

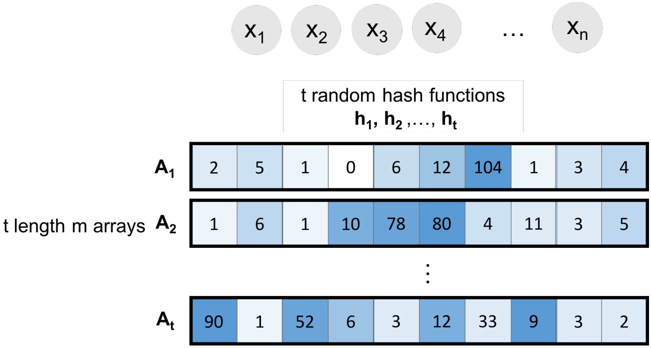

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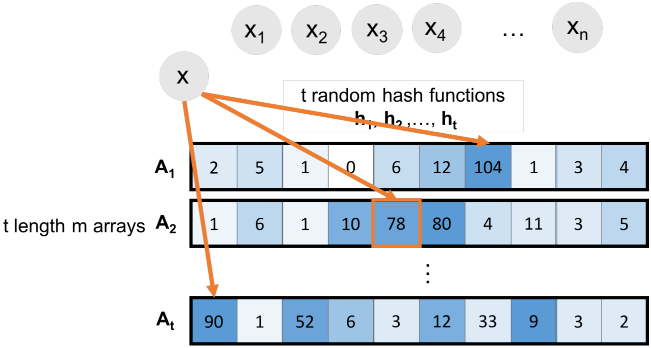

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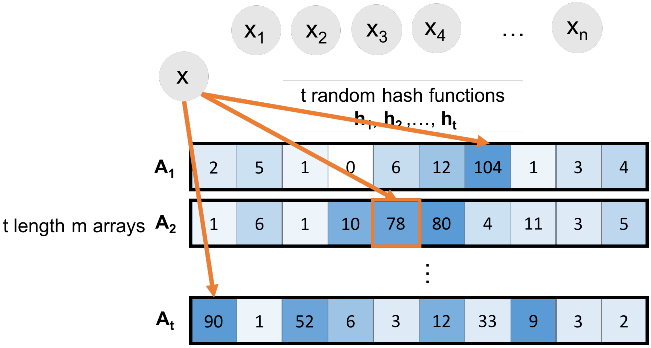

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Why min instead of mean or median?

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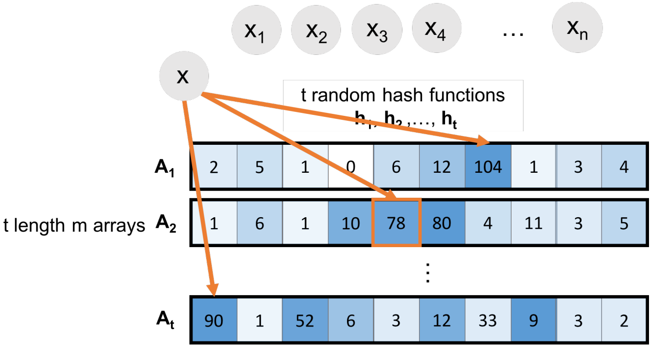
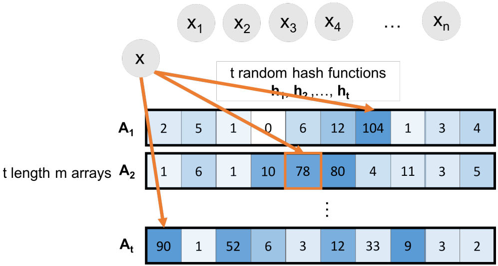


Estimate $f(x)$ with $\tilde{f}(x) = \min_{i \in [t]} A_i[\mathbf{h}_i(x)]$. (count-min sketch)

Why min instead of mean or median? The minimum estimate is always the most accurate since they are all overestimates of the true frequency!



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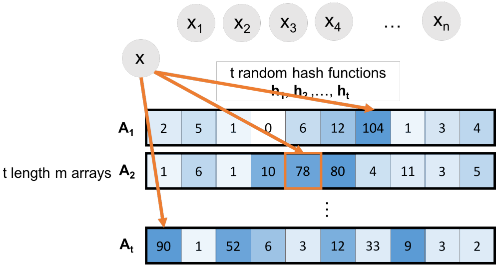
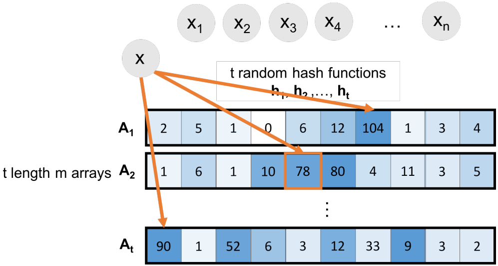


Estimate $f(x)$ by $\tilde{f}(x) = \min_{i\in[t]} A_i[\mathbf{h}_i(x)]$

- For every x and $i \in [t]$, we know that for $m = \frac{2k}{\epsilon}$, with probability $\geq 1/2$:

$$f(x) \leq A_i[\mathbf{h}_i(x)] \leq f(x) + \frac{\epsilon n}{k}.$$

COUNT-MIN SKETCH ANALYSIS



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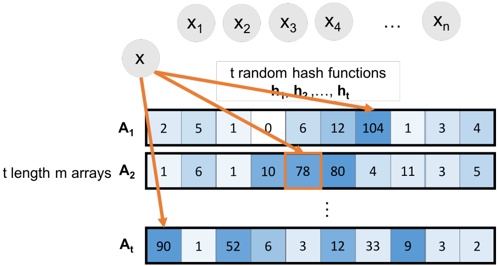

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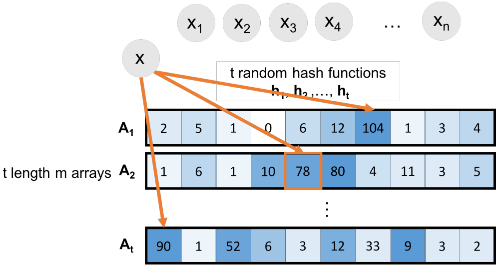

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- What is $\Pr[f(x) \leq \tilde{f}(x) \leq f(x) + \frac{\epsilon n}{k}]$? $1 - 1/2^t$.
- To get a good estimate with probability $\geq 1 - \delta$, set $t = \log(1/\delta)$.

Upshot: Count-min sketch lets us estimate the frequency of every item in a stream up to error $\frac{\epsilon n}{k}$ with probability $\geq 1 - \delta$ in $O\left(\log(1/\delta) \cdot k/\epsilon\right)$ space.

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- How should we set δ if we want a good estimate for all items at once, with 99% probability? $\delta = 0.01/|U|$ ensures

$$\Pr[\text{there exists } x \in U \text{ with a bad estimate}]$$
$$\leq \sum_{x \in U} \Pr[\text{estimate for } x \text{ is bad}] \leq \sum_{x \in U} 0.01/|U| = 0.01$$

IDENTIFYING FREQUENT ELEMENTS

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One approach:

- Maintain a set F while processing the stream:
- At step i:
  - Add ith stream element to F if it's estimated frequency is $\geq i/k$ and it isn't already in F.
  - Remove any element from F whose estimated frequency is $< i/k$.
- Store at most k items at once and have all items with frequency $\geq n/k$ stored at the end of the stream.

Questions on Frequent Elements?