CMPSCI 611 FALL '09: HOMEWORK 1
DUE 2:30 PM, SEPTEMBER 29TH

Rules:

- Collaborating with at most two other students is allowed but answers must be written independently. Solutions must be typed (ideally in Latex). Please write with whom you collaborated on your solutions. Unfortunately, you're not allowed to use material from the net or talk about the homework with anybody outside your collaboration group (aside from the lecturer or TA of course.)
- Solutions are due before 2:31 pm (according to the clock in the classroom) on the due date. Marks for solutions that are returned up to 24 hours late will be reduced by 25%. After that no credit will be given. Email `mcgregor@cs.umass.edu` if your homework is going to be late.
- To get full marks, answers must be sufficiently detailed, supported with rigorous mathematical proofs, and be clearly explained.

Question 1 (15 marks). *a) Take the following list of functions and arrange them in ascending order of growth rate. That is, if function $g(n)$ immediately follows function $f(n)$ in your list, then it should be the case that $f(n)$ is $O(g(n))$.*

$$g_1(n) = 2^{\sqrt{\log_2 n}},\ g_2(n) = 2^n,\ g_3(n) = n(\log_2 n)^3,\ g_4(n) = n^{4/3},$$
$$g_5(n) = n^{\log_2 n},\ g_6(n) = 2^{2^n},\ g_7(n) = (n^{1/\log_2 n})^3,\ g_8(n) = \pi^n$$

b) Assume that you have functions f and g such that $f(n)$ is $O(g(n))$. For each of the following statements, either prove that it is correct or provide a counter-example:

(1) $\log_2 f(n)$ *is* $O(\log_2 g(n))$
(2) $2^{f(n)}$ *is* $O(2^{g(n)})$
(3) $f(n)^2$ *is* $O(g(n)^2)$

Question 2 (15 marks). *a) Given a sequence of n distinct numbers $A = a_1, a_2, \ldots, a_n$, we say that pair a_i and a_j are* inverted *if $i < j$ but $a_i > a_j$. Let $\nu(A)$ be the number of pairs that are inverted. For example, if A is sorted in increasing order then $\nu(A) = 0$ whereas, if A is sorted in decreasing order then $\nu(A) = n(n-1)/2$. Design a divide and conquer algorithm that computes $\nu(A)$ and prove that it takes $\Theta(n \log n)$ time. Assume that the sequence A is given in an array such that we can test whether $a_i < a_j$ or $a_j < a_i$ in unit time for any $i, j \in [n] = \{1, 2, \ldots, n\}$. [Hint: Revisit Merge-Sort].*

b) We say a_i and a_j are very inverted *if $i < j$ but $a_i > 2a_j$. Let $\mu(A)$ be the number of pairs that are very inverted. Design a divide and conquer algorithm that computes $\mu(A)$ and prove that it takes $\Theta(n \log n)$ time.*

Question 3 (15 marks). *You are given a set of n non-vertical lines in the plane, where the i-th line, ℓ_i, is described by two real numbers a_i and b_i such that ℓ_i consists of all points (x, y) that satisfy $y = a_i x + b_i$. We say that a line ℓ dominates at t, if the y-coordinate of ℓ is larger at $x = t$ than the y-coordinate at any other line at $x = t$. We say that line ℓ is visible if there is some t at which ℓ dominates.*

Design an $\Theta(n \log n)$ time algorithm that outputs a list of all the lines that are visible. Prove that the algorithm is correct and that the running time is as claimed. You may assume that no three lines all meet at a single point.

Question 4 (5 marks)**.** *Let* $\omega_n = e^{2\pi i/n} = \cos\frac{2\pi}{n} + i\sin\frac{2\pi}{n}$*. Define*

$$A = \begin{pmatrix} 1 & 1 & 1 & 1 & \dots & 1 \\ 1 & \omega_n & \omega_n^2 & \omega_n^3 & \dots & \omega_n^{n-1} \\ 1 & \omega_n^2 & \omega_n^4 & \omega_n^6 & \dots & \omega_n^{2(n-1)} \\ \vdots & \vdots & \vdots & \vdots & & \vdots \\ 1 & \omega_n^{n-1} & \omega_n^{2(n-1)} & \omega_n^{3(n-1)} & \dots & \omega_n^{(n-1)(n-1)} \end{pmatrix}$$

and

$$B = \frac{1}{n} \begin{pmatrix} 1 & 1 & 1 & 1 & \dots & 1 \\ 1 & \omega_n^{-1} & \omega_n^{-2} & \omega_n^{-3} & \dots & \omega_n^{-(n-1)} \\ 1 & \omega_n^{-2} & \omega_n^{-4} & \omega_n^{-6} & \dots & \omega_n^{-2(n-1)} \\ \vdots & \vdots & \vdots & \vdots & & \vdots \\ 1 & \omega_n^{-(n-1)} & \omega_n^{-2(n-1)} & \omega_n^{-3(n-1)} & \dots & \omega_n^{-(n-1)(n-1)} \end{pmatrix}.$$

Prove that $AB = BA = I$ *where* I *is the identity matrix.*