CMPSCI 611: Advanced Algorithms

Lecture 5: Greedy Algorithms and Matroids

Andrew McGregor

Last Compiled: September 22, 2009

Outline

Recap of Generic Problem

Matroids

Alternative Characterization via “Cardinality Property”

Recap from Last Time

Definition

A *subset system* $S = (E, \mathcal{I})$ is a finite set E with a collection $\mathcal{I}$ of subsets E such that:

$$\text{if } i \in \mathcal{I} \text{ and } i' \subset i \text{ then } i' \in \mathcal{I}$$

i.e., "$\mathcal{I}$ is closed under inclusion"

Problem Given a subset system $S = (E, \mathcal{I})$ and weight function $w : E \to \mathbb{R}^+$, find $i \in \mathcal{I}$ such that $w(i) = \sum_{e \in i} w(e)$ is maximized.

Algorithm (Greedy)

1. $i = \emptyset$
2. *Sort elements of E by non-increasing weight*
3. *For each $e \in E$: If $i + e \in \mathcal{I}$ then $i = i + e$*

Outline

Recap of Generic Problem

Matroids

Alternative Characterization via “Cardinality Property”

Matroid Definition and Theorem

Definition

A matroid is a subset system $M = (E, \mathcal{I})$ that satisfies the exchange property: if $i, i' \in \mathcal{I}$ such that $|i| < |i'|$, then there exists $e \in i' - i$ with $i + e \in \mathcal{I}$

Theorem

For any subset system $(E, \mathcal{I})$, the greedy algorithm solves the optimization problem for $(E, \mathcal{I})$ if and only if $(E, \mathcal{I})$ is a matroid.

Definition

$i \in \mathcal{I}$ is maximal is there doesn't exist e with $i + e \in \mathcal{I}$. It is maximum if there doesn't exist $j \in \mathcal{I}$ with $w(j) > w(i)$.

Note that the greedy algorithm produces a solution that is always maximal and a maximum solution is always maximal.

Matroid implies Greedy Algorithm is Optimal (1/2)

- Proof by contradiction: Assume $(E, \mathcal{I})$ is a matroid and let

  $$\begin{aligned} \text{greedy solution:} \quad & i = \{e_1, e_2, \ldots, e_k\} \\ \text{optimal solution:} \quad & j = \{e'_1, e'_2, \ldots, e'_{k'}\} \end{aligned}$$

  where $w(j) > w(i)$.
- Can deduce $k = k'$ by the exchange property.
- Assuming our numbering of $e \in i$, $e' \in j$ satisfies:

  $$\begin{aligned} w(e_1) \geq w(e_2) \geq \ldots \geq w(e_k) \\ w(e'_1) \geq w(e'_2) \geq \ldots \geq w(e'_k) \end{aligned}$$

- Exists s with $w(e'_s) > w(e_s)$. Consider smallest such s.
- Consider subsets of i and j:

  $$\alpha = \{e_1, e_2, \ldots, e_{s-1}\} \quad \text{and} \quad \beta = \{e'_1, e'_2, \ldots, e'_s\}$$

Matroid implies Greedy Algorithm is Optimal (2/2)

- Consider subsets of i and j:

$$\alpha = \{e_1, e_2, \ldots, e_{s-1}\} \quad \text{and} \quad \beta = \{e'_1, e'_2, \ldots, e'_s\}$$

- By the exchange property there exists

$$e'_t \in \beta - \alpha \;\text{ with }\; \alpha + e'_t \in \mathcal{I}$$

- But then $w(e'_t) \geq w(e'_s)$ and hence $w(e'_t) > w(e_s)$
- Greedy algorithm shouldn't have picked e_s! Contradiction.

Optimality of Greedy Algorithm implies $(E, \mathcal{I})$ is a Matroid

- Sufficient to show that if $(E, \mathcal{I})$ is not a matroid then greedy doesn't necessary work.
- $(E, \mathcal{I})$ not a matroid implies that

$$\exists\ i, i' \in \mathcal{I} \text{ such that } |i| < |i'| \text{ and } \nexists\ e \in i' \text{ with } i + e \in \mathcal{I}$$

- Let $m = |i|$ and $n = |E|$. Define weight function to "trip up" greedy algorithm:

$$w(e) = \begin{cases} m+2 & \text{if } e \in i \\ m+1 & \text{if } e \in i' - i \\ 1/(2n) & \text{otherwise} \end{cases}$$

- Greedy algorithm returns i with weight at most $(m+2)m$ $+1/2$ but a better solution is i' with weight at least $(m+1)^2$

Outline

Recap of Generic Problem

Matroids

Alternative Characterization via “Cardinality Property”

Alternative Characterization of Matroid

Definition

For $A \subseteq E$, we say $i \in \mathcal{I}$ is a maximal subset of A if $i \subseteq A$ and there doesn't exist $e \in A$ such that $i + e \in \mathcal{I}$.

Theorem (Cardinality Theorem)

A subset system $(E, \mathcal{I})$ is a matroid if and only:

$$\forall A \subseteq E,\ \textit{if } i, i' \in \mathcal{I} \textit{ are maximal subsets of } A \textit{ then } |i| = |i'|$$

We'll call this the cardinality property.

Matroid Property implies Cardinality Property

- Let i, i' be maximal subsets of $A \subseteq E$. Need to show $|i| = |i'|$
- If $|i'| > |i|$, the exchange property implies

$$\exists\ e \in i' - i \text{ such that } i + e \in \mathcal{I}$$

- Thus i was not maximal in A. Contradiction!

Cardinality Property implies Matroid Property

- Suffices to show that $(E, \mathcal{I})$ not a matroid implies there exists A and $i, j \in \mathcal{I}$ such that $|i| \neq |j|$ and i, j are maximal in A
- $(E, \mathcal{I})$ not a matroid implies that

$$\exists\ i, i' \in \mathcal{I} \text{ such that } |i| < |i'| \text{ and } \nexists\ e \in i' \text{ with } i + e \in \mathcal{I}$$

- Define $A = i \cup i'$ and note that i is maximal in A.
- Either i' is maximal in A or there exists $j \in \mathcal{I}$ such that $i' \subseteq j$ and j is maximal in A.
- But $|j| \geq |i| + 1$ as required.

For Next Time…

- Finish reading upto section 3.4 of the notes.
- And make sure you've started the homework!