CMPSCI 611: Advanced Algorithms

Lecture 15: Tail Inequalities

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Outline

Markov and Chebyshev

Lazy Select

Markov Inequality

Theorem (Markov)

Let Y be a non-negative random variable and let $\mu_Y = \mathbb{E}[Y]$. Then, for all $t > 0$, $\mathbb{P}[Y \geq t\mu_Y] \leq 1/t$.

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- Note that $Z \leq Y/(t\mu_Y)$.

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- $\mathbb{P}[Y \geq t\mu_Y] = \mathbb{E}[Z] \leq \mathbb{E}[Y/(t\mu_Y)] = 1/t$

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Chebyshev Inequality

Definition

Variance of X is $\mathbb{V}[X] = \mathbb{E}\left[(X - \mathbb{E}[X])^2\right] = \mathbb{E}\left[X^2\right] - \mathbb{E}[X]^2$ and standard deviation: $\sigma_X = \sqrt{\mathbb{V}[X]}$.

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Theorem (Chebyshev)

Let X be a random variable with expectation μ_X and standard deviation σ_X. Then for $t > 0$, $\mathbb{P}[|X - \mu_X| \geq t\sigma_X] \leq 1/t^2$.

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- Note that $\mathbb{P}[|X - \mu_X| \geq t\sigma_X] = \mathbb{P}\left[(X - \mu_X)^2 \geq t^2\sigma_X^2\right]$

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- Let $Y = (X - \mu_X)^2$ and note $\mathbb{E}[Y] = \sigma_X^2$
- Use Markov's inequality to show $\mathbb{P}\left[Y \geq t^2\mathbb{E}[Y]\right] \leq 1/t^2$

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$$\mathrm{rank}_R(a) = kn^{-1/4} - \sqrt{n} \text{ and } \mathrm{rank}_R(b) = kn^{-1/4} + \sqrt{n}$$

where $\mathrm{rank}_X(x) = t$ if x is the t-th smallest element in X.

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5. *Return $(k - \text{rank}_S(a) + 1)$-th smallest element from P*

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- $O(n)$ steps to compute $\text{rank}_S(a)$ and $\text{rank}_S(b)$ in S.
- $O(n^{3/4} \log n)$ steps to sort P and select element.

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- Only three ways in which we fail and we'll show
  1. $\mathbb{P}[k < \text{rank}_S(a)] \leq O(n^{-1/4})$
  2. $\mathbb{P}[k > \text{rank}_S(b)] \leq O(n^{-1/4})$
  3. $\mathbb{P}\left[|P| \geq 4n^{3/4}\right] \leq O(n^{-1/4})$

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  3. $\mathbb{P}\left[|P| \geq 4n^{3/4}\right] \leq O(n^{-1/4})$
- Result follows by the Union Bound

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Proof.

- Let u be the k-th smallest element in S
- Consider choosing R: Let $X_i = 1$ if i-th sample is $\leq u$ and $X_i = 0$ otherwise. $\mathbb{P}[X_i = 1] = k/n$ and $\mathbb{P}[X_i = 0] = 1 - k/n$

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- $k < \text{rank}_S(a)$ implies $X < kn^{-1/4} - \sqrt{n}$
- X has binomial distribution:

$$\mathbb{E}[X] = kn^{-1/4} \quad \text{and} \quad \mathbb{V}[X] = n^{3/4}(k/n)(1 - k/n) = n^{3/4}/4$$

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- Apply Chebyshev bound: $\mathbb{P}\left[X < kn^{-1/4} - \sqrt{n}\right]$ is at most

$$\mathbb{P}\left[|X - \mathbb{E}\left[X\right]| > \sqrt{n}\right] \leq \mathbb{P}\left[|X - \mathbb{E}\left[X\right]| > 2n^{1/8}\sigma_X\right] = O(n^{-1/4})$$

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- Apply union bound.

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