NAME: ____________________________

CMPSCI 611
Advanced Algorithms
Final Exam Fall 2012

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DIRECTIONS:

- Do not turn over the page until you are told to do so.
- This is a *closed book exam.* No communicating with other students, or looking at notes, or using electronic devices. You may ask the professor to clarify the meaning of a question but do so in a way that causes minimal disruption.
- If you finish early, you may leave early but do so as quietly as possible. The exam script should be given to the professor.
- There are five questions. All carry the same number of points but some questions may be easier than others. Don't spend too long on a problem if you're stuck – you may find that there are other easier questions.
- The front and back of the pages can be used for solutions. There is also a blank page at the end that can be used. If you are using these pages, clearly indicate which question you're answering.
- The exam will finish at 12:30 pm.
- Good luck!

1	/10
2	/10
3	/10
4	/10
5	/10
Total	/50

Question 1 (True or False): For each of the following statements, indicate whether the statement is true or false by circling the appropriate option. It is not necessary to show working.

1. Given an unweighted graph, it is possible to find the length of the shortest path between every pair of vertices in polynomial time.

   - TRUE
   - FALSE

2. For any satisfiable instance ϕ of 3-SAT with more than one clause, a random assignment of the variables will satisfy ϕ with probability at least $7/8$.

   - TRUE
   - FALSE

3. If $T(1) = 1$ and $T(n) = 2T(n/2) + n$ then $T(n) = \Theta(n \log n)$.

   - TRUE
   - FALSE

4. For any graph $G = (V, E)$, if $V' \subset V$ is a vertex cover then $V \setminus V'$ is an independent set.

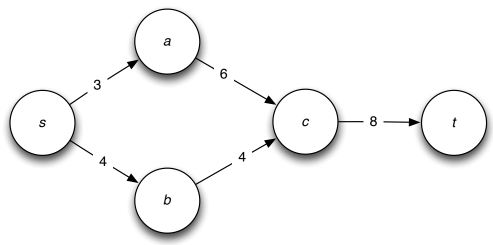
   - TRUE
   - FALSE

5. For any random variable X which is never negative, $\mathbb{P}\left[X < 10 \cdot \mathbb{E}\left[X\right]\right] \geq 9/10$.

   - TRUE
   - FALSE

Question 2 (Flows and Linear Programming):

1. What is the maximum s-t flow in the following network where labels on the edges indicate the capacities of the edges?



2. State the Max-Flow/Min-Cut theorem.

3. TRUE or FALSE: There exists a polynomial time algorithm for solving linear programs.

   - TRUE
   - FALSE

In the next two parts of the question, we consider the following linear program.

$$\begin{array}{rl} \text{maximize} & 2x_1 + x_2 \\ \text{subject to} & x_1 + x_2 \leq 3 \ , \ x_1 \leq 2 \ , \ x_2 \leq 2 \ , \ x_1, x_2 \geq 0 \end{array}$$

4. Draw the feasible region for the linear program. What is the optimal value?

5. Write the dual of the above linear program. What is the optimal value of the dual?

Question 3 (Matchings): Consider a complete bipartite graph $G = (L \cup R, E)$ with nodes $L = \{l_1, \ldots, l_n\}$, $R = \{r_1, \ldots, r_n\}$, and edges $E = \{(l_i, r_j) : i, j \in \{1, 2, \ldots, n\}\}$. Suppose each edge $e \in E$ has some weight $w_e \geq 0$. The minimum weight matching problem is to find a matching M with n edges such that $\sum_{e \in M} w_e$ is minimized.

1. Prove that the minimum weight matching problem can be solved in polynomial time. You may use the fact that the maximum weight matching problem (i.e., find a matching that maximizes $\sum_{e \in M} w_e$) can be solved in polynomial time.

2. Suppose each node v has a weight w_v such that $w_{l_1} < w_{l_2} < \ldots < w_{l_n}$ and $w_{r_1} < w_{r_2} < \ldots < w_{r_n}$. If the weight of an edge $e = (l, r)$ is defined as $w_e = |w_r - w_l|$, what is the weight of the minimum weight matching? Prove your answer is correct. **Hint**: What could you do if a matching contained edges (l_i, r_k) and (l_j, r_ℓ) for some $i < j$ and $k > \ell$?

Question 4 (Verifying Matrix Multiplication): In this question we will design a randomized algorithm for checking matrix multiplication. You may use the following result without proof: If $Q(x_1, \ldots, x_n)$ is a non-zero polynomial with degree at most d and if $r_1, \ldots, r_n$ are chosen randomly from $\{0, 1, 2, \ldots, s-1\}$, then $\mathbb{P}[Q(r_1, \ldots, r_n) = 0] \leq d/s$.

1. Give an example of Q with $n = 1, d = 2, s = 3$ such that the above inequality is tight.

2. Your friend claims that when you multiply two $n \times n$ matrices A and B, the resulting matrix is C. To check her answer, you randomly choose $r_1, \ldots, r_n$ such that each r_i is equally likely to be 0 or 1 and all r_i are independent. This defines vector $r = (r_1, r_2, \ldots, r_n)^T$.
   (a) If $AB \neq C$, prove that $\mathbb{P}[ABr \neq Cr] \geq 1/2$.

   (b) Give upper and lower bounds on the smallest time to compute ABr?

3. Design and prove the correctness of an efficient randomized algorithm that, given input A, B, C and a positive parameter $\delta < 1/2$, will output "same" if $AB = C$ and will output "different" with probability at least $1 - \delta$ if $AB \neq C$. The more efficient your algorithm, the more marks you receive. You may assume the results in Part 2.

Question 5 (Coloring Graphs): A t-coloring of a graph $G = (V, E)$ is a labeling $f : V \to \{1, 2, \ldots, t\}$ of the vertices such that if $(u, v) \in E$ then $f(u) \neq f(v)$, i.e., adjacent nodes receive different labels. We say a graph G is t-colorable if there exists a t-coloring for G.

1. Design a polynomial time algorithm that determines whether a graph is 2-colorable. Prove the approximation ratio. **Hint:** Consider performing a breadth first search.

2. Let D be a maximum degree of a graph. Design a polynomial time algorithm that $(D + 1)/3$ approximates the minimum value of t such that the graph is t-colorable. Prove the approximation ratio.