NAME: ____________________________

CMPSCI 611
Advanced Algorithms
First Midterm Exam Spring 2024

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DIRECTIONS:

- Do not turn over the page until you are told to do so.
- This is a *closed book exam.* No communicating with other students, or looking at notes, or using electronic devices. You may ask the professor or TA to clarify the meaning of a question but do so in a way that causes minimal disruption.
- If you finish early, you may leave early but do so as quietly as possible. The exam script should be given to the professor.
- There are four questions. All carry the same number of marks but some questions may be easier than others. Don't spend too long on a problem if you're stuck – you may find that there are other easier questions.
- Write all answers in the space provided. You may use the back of pages for rough work.

1	/8
2	/8
3	/8
4	/8
Total	/32

Question 1. For each of the following statements, indicate whether they are TRUE or FALSE by circling the appropriate answer. No justification is required. Each part is worth 2 points.

1. $5n^2 + 10 = O(n^2)$.

   TRUE FALSE

2. A minimum spanning tree has no cycles.

   TRUE FALSE

3. Every subset system satisfies the cardinality property.

   TRUE FALSE

4. Let G be a connected, undirected graph with n nodes. If all edges have length 1, then the shortest path between any two nodes is at most $n - 1$.

   TRUE FALSE

Question 2. No justification required for your answers. Each part is worth 2 points.

1. We want to sort a list by performing "prefix reversals" operations, i.e., at each step we specify a value k and then reverse the ordering of the first k entries of the list. What is the minimum number of prefix reversals needed to sort $[7, 8, 6, 5, 1, 2, 3, 4]$. **Hint:** We know from homework that it is at most 2×8 but the answer here is less than that.

2. Let $E = \{e_1, e_2, e_3\}$ and $\mathcal{I} = \{\{\}, \{e_1\}, \{e_1, e_2\}, \{e_3\}\}$. What set in $\mathcal{I}$ should you remove if you wanted $(E, \mathcal{I})$ to be a subset system that satisfied the exchange property.

3. Let M be the adjacency matrix of the graph with three nodes v_1, v_2, v_3 and two edges (v_1, v_2) and (v_2, v_3). Write out M^2, i.e., the square of the adjacency matrix.

4. If $T(n) = T(n-1) + n^2$ for $n > 1$ and $T(1) = 1$, then $T(n) = \Theta(n^x)$ for what value of x?

Question 3. A company wants to buy some of the existing n gas stations along a highway. For $i \in \{1, 2, \ldots, n\}$, let p_i be the profit they will make from owning the ith gas station. You may assume all p_i values are positive. A state law says forbids the same company from owning two gas stations that are next to each other, e.g., if the company owns the 5th gas station, they can't also own the 4th or 6th gas station.

1. (2 points) Suppose $n = 5$ and $p_1 = 5, p_2 = 3, p_3 = 2, p_4 = 4$ and $p_5 = 1$. What is the maximum possible profit that can be achieved?

2. (2 points) Consider the algorithm that, at each step, buys the gas station that has max profit amongst those that are not next to a gas station that was already purchased. Suppose $n = 3$. Give examples of values for p_1, p_2, p_3 such that this algorithm would *not* return the best solution.

3. (4 points) In this part you should not make any assumptions about n or the profits. Design a dynamic programming algorithm that finds the maximum total profit that is possible. Analyze the running time and justify correctness.

Question 4. We now consider a simpler (but less efficient) algorithm for multiplying two polynomials than the one discussed in class. Assume n is a power of two. The input is the coefficients of the following polynomials $a(x) = a_0 + a_1 x + \ldots + a_{n-1}x^{n-1}$ and $b(x) = b_0 + b_1 x + \ldots + b_{n-1}x^{n-1}$ and the goal is to compute the coefficients of the polynomial $c(x) = a(x)b(x)$. Define

$$\begin{aligned} a_L(x) &= a_0 + a_1 x + \ldots + a_{n/2-1}x^{n/2-1} \\ a_H(x) &= a_{n/2} + a_{n/2+1}x + \ldots + a_{n-1}x^{n/2-1} \\ b_L(x) &= b_0 + b_1 x + \ldots + b_{n/2-1}x^{n/2-1} \\ b_H(x) &= b_{n/2} + b_{n/2+1}x + \ldots + b_{n-1}x^{n/2-1} \end{aligned}$$

and note that $a(x) = a_L(x) + x^{n/2}a_H(x)$ and $b(x) = b_L(x) + x^{n/2}b_H(x)$.

1. (2 points) Write $c(x)$ in terms of $a_L(x), a_H(x), b_L(x), b_H(x), x^{n/2}$, and x^n. Your answer should be written as the sum of four terms.

2. (2 points) Write $c(x)$ in terms of $p_1(x), p_2(x), p_3(x)$, $x^{n/2}$, and x^n where

$$p_1(x) = a_L(x)b_L(x) \qquad p_2(x) = a_H(x)b_H(x)$$

$$p_3(x) = (a_L(x) + a_H(x))(b_L(x) + b_H(x)) - p_1(x) - p_2(x)$$

Hint: First simplify $p_3(x)$.

3. (4 points) Write a divide and conquer algorithm for computing $c(x)$ based on your answers above. Analyze the running time and justify correctness.