NAME: ____________________________

COMPSCI 611
Advanced Algorithms
Second Midterm Exam Fall 2017

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DIRECTIONS:

- Do not turn over the page until you are told to do so.
- This is a *closed book exam.* No communicating with other students, or looking at notes, or using electronic devices. You may ask the professor or TA to clarify the meaning of a question but do so in a way that causes minimal disruption.
- If you finish early, you may leave early but do so as quietly as possible. The exam script should be given to the professor.
- There are five questions. All carry the same number of marks but some questions may be easier than others. Don't spend too long on a problem if you're stuck – you may find that there are other easier questions.
- The front and back of the pages can be used for solutions. There are also a blank page at the end that can be used. If you are using these pages, clearly indicate which question you're answering. Further paper can be requested if required. However, the best answers are those that are clear and concise (and of course correct).

1	/10
2	/10
3	/10
4	/10
5	/8+2
Total	/48+2

Question 1 (The True or False Question). For each of the following statements, say whether they are TRUE or FALSE. No justification is required.

1. If the expected running time of a randomized algorithm is 10 minutes, then the probability it takes more than 50 minutes is at most 1/5.
   Answer: TRUE.

2. Every tree with n nodes has $n-1$ minimum cuts.
   Answer: TRUE.

3. Suppose the maximum flow in an instance of network flow is 10. If we double the capacity of every edge then the maximum flow is now 20.
   Answer: TRUE.

4. If a polynomial $f(x, y)$ has degree d then there are at most d pairs of values (a, b) such that $f(a, b) = 0$.
   Answer: FALSE. Consider $f(x, y) = x - y$.

5. If we run (randomized) Quicksort on a set of n distinct values, the probability that the smallest value gets compared directly to the second smallest value is 1/2.
   Answer: FALSE. The probability would be 1.

Question 2 (Matching). 1. Consider the following undirected, unweighted graph.

Size of maximum matching $=$ 4

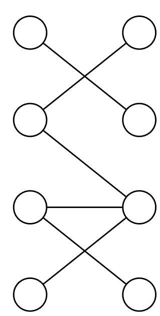
Size of minimum vertex cover $=$ 4

Number of edges $=$ 6

Size of the maximum cut $=$ 6

Size of the minimum cut $=$ 0

Size of the minimum degree $=$ 1



Hint: The above graph is bipartite and you might want to use that fact above.

2. Recall, a matching is *perfect* if every node is an endpoint of some edge in the matching. Find a graph where every vertex has degree 2 that does not have a perfect matching.
   Answer: A triangle. Or in general, any cycle with an odd number of nodes.

3. Let $G = (L \cup R, E)$ be a bipartite graph where L and R each have 120 nodes and every node has degree 4. What is the size of the maximum matching? Justify your answer. **Hint:** If $|\Gamma(U)| \geq |U|$ for every $U \subseteq L$ where $\Gamma(U) = \{v \in R : (u, v) \in E\}$, you proved in the homework that G has a perfect matching.
   Answer: Size of the maximum matching G is 120, i.e., there's a perfect matching. This follows because for any $U \subset L$, $|\Gamma(U)|$ has at least $4|U|$ incident edges. But because the degree of any node in $\Gamma(U)$ is at most 4, this means $4\Gamma(U) \geq 4|U|$, i.e., $|\Gamma(U)| \geq |U|$.

Question 3. Suppose that each time a randomized algorithm is run, it returns a value in the set $\{0, 1, 2, \ldots, 6\}$. The probability of each value is not necessarily the same. The expected value returned is 3 and this is the correct value for the problem being solved. Let X_i be the value returned on the ith run of the algorithm and $X_1, X_2, X_3, \ldots$ are all independent.

1. What is the maximum value of the probability $P(X_1 = 6)$ given the information above?
   Answer: $P(X_1 = 6) = P(X_1 \geq 6) \leq E(X_1)/6 = 1/2$.

2. Why is $\text{Var}(X_1) = E\left(|X_1 - E(X_1)|^2\right)$ at most 9 given the information above?
   Answer: Since $0 \leq X_1 \leq 6$ and $E(X_1) = 3$, $|X_1 - E(X_1)|^2 \leq 9$. The expectation of a random variable whose maximum value is 9 is also at most 9.

3. Suppose the algorithm is run 50 times. Use the Chebyshev bound to upper bound the probability that the average value returned is *not* between 2.5 and 3.5? **Hint:** Analyze $X = \sum_{i=1}^{50} X_i$ and use the fact $\text{Var}(X) = \sum_{i=1}^{50} \text{Var}(X_i)$. You may assume $\text{Var}(X_i) \leq 9$.
   Answer: The average is $X/50$. So $P(X/50 > 3.5 \text{ or } X/50 < 2.5)$ equals

   $$P(|X/50 - 3| \geq 0.5) = P(|X - 150| \geq 25) \leq \frac{\text{Var}(X)}{25^2} \leq \frac{9 \times 50}{25^2} = \frac{18}{25}\,,$$

   where we used the fact that $E(X) = 150$ by linearity of expectation.

For the last two parts, suppose the probability of returning the correct answer is exactly 4/5 and that you run the algorithm 3 times.

4. Probability the algorithm never returned the correct answer =
   Answer: $(1 - 4/5)^3 = 1/125$.

5. Probability the correct answer is strictly more common than any other answer =
   Answer: $P(\text{ all correct }) + P(\text{ exactly two correct}) = (4/5)^3 + 3(4/5)^2(1/5) = 112/125$.

Question 4. You are holding a one meter wooden rod horizontally and you need to break it into n pieces. There are $n-1$ *breakpoints*, i.e., positions where the rod can be snapped. Measured from the left end of the rod, these are at positions $0 < x_1 < x_2 < \ldots < x_{n-1} < 1$. You can snap the rod at these $n-1$ positions in any order but when you snap the rod at a breakpoint, the effort required is equal to the current length of the portion of the rod that contains that breakpoint.

E.g., if $n = 4$ and the break points are $x_1 = 0.3,\ x_2 = 0.4,\ x_3 = 0.6$, then snapping the breakpoints in the order x_2 (leaving a piece of length 0.4 and 0.6), then x_3 (splitting the piece of length 0.6 into two), then x_1 (splitting the piece of length 0.4 into 2) requires total effort $1 + 0.6 + 0.4 = 2$

1. If $n = 4$ and the break points are $x_1 = 0.3,\ x_2 = 0.4,\ x_3 = 0.6$, what is the total effort if you snap the breakpoints in the order x_3, then x_1, then x_2?

   Answer: The cost is $1 + 0.6 + 0.3 = 1.9$.

2. Let $0 < x_1 < x_2 < \ldots < x_{n-1} < 1$ be arbitrary. Design a *dynamic program* to find the minimum effort required to break rod into n pieces. **Hint:** Let $D[i, j]$ be the minimum effort that would be required to break the portion of the rod from x_i and x_j into $j - i$ pieces assuming it had already been broken at x_i and x_j. Define $x_0 = 0$ and $x_n = 1$.

   (a) Describe your dynamic programming algorithm.
   Answer:

   i. For $i = 0$ to $n - 1$, let $D[i, i + 1] = 0$.
   ii. For $i = 0$ to $n - 2$, let $D[i, i + 2] = x_{i+2} - x_i$.
   iii. For $k = 3$ to n: For $i = 0$ to $n - k$, let

   $$D[i, i + k] = x_{i+k} - x_i + \min_{i<j<i+k} (D[i, j] + D[j, i + k])$$

   iv. Return $D[0, n]$

   (b) Briefly justify the correctness of your algorithm.
   Answer: $D[i, i + k] = x_{i+k} - x_i + \min_{i<j<i+k}(D[i, j] + D[j, i + k])$ because if we break it at x_j first we pay $x_{i+k} - x_i$ and then are left with the portion x_i to x_j (which has cost $D[i, j]$) and the portion x_j to x_{i+k} (which has cost $D[j, i+k]$. Note the $D[i, j]$ and $D[j, i + k]$ will have already been computed because of the way we set up the loops.

   (c) What is the running time?
   Answer: The running time is $O(n^3)$ because there are $O(n^2)$ values $D[i, j]$ to be computed and each is computed in $O(n)$ time.

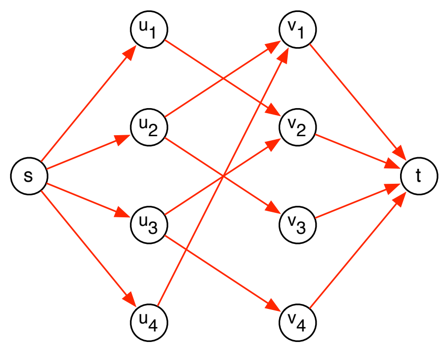
Question 5. In this question, we suppose that there are four students who are going to buy presents for each other in such a way that each student gives exactly one present and each student receives exactly one present. Suppose that each student only feels comfortable buying a present for students amongst some subset of the other students: let S_i be the set of students for whom the ith student is comfortable buying a present. In the first three parts of this question, you should assume that S_i does not include i, i.e., the ith student doesn't feel comfortable buying a present for themselves. We say that *present buying is possible* if it is possible for everyone to buy exactly one present (for someone they are comfortable buying a present for) and everyone receives exactly one present.

1. Who should students buy a present for if $S_1 = \{2\}, S_2 = \{1, 3\}, S_3 = \{2, 4\}, S_4 = \{1\}$.

   student 1 buys present for ... 2   student 2 buys present for ... 3

   student 3 buys present for ... 4   student 4 buys present for ... 1

2. Consider the directed graph G_{S_1,S_2,S_3,S_4} with nodes $s, u_1, u_2, u_3, u_4, v_1, v_2, v_3, v_4, t$ where there is a directed edge from s to u_1, u_2, u_3, u_4, from v_1, v_2, v_3, v_4 to t, and from u_i to v_j iff $j \in S_i$. Draw the graph if $S_1 = \{2\}, S_2 = \{1, 3\}, S_3 = \{2, 4\}, S_4 = \{1\}$ and compute the maximum flow on the assumption that every edge has capacity one.



   Size of maximum flow = 4

3. Let S_1, S_2, S_3, S_4 be arbitrary. Briefly explain why if the maximum s-t flow in G_{S_1,S_2,S_3,S_4} is 4 then present buying is possible.

   Answer: A flow of size 4 implies edges used in the flow between the u nodes and the v nodes is a perfect matching, i.e., each u_i has exactly one outgoing edge and each v_j has exactly one incoming edge. This means that if student i buys a present for student j, every student has given one present (and that student was in the appropriate set) and every student has received one present.

In the rest of the question, suppose a) there are now 70 students and b) that everyone is comfortable buying anyone, including themselves, a present, i.e., $S_1 = S_2 = \ldots = S_{70} = \{1, \ldots, 70\}$. The way they choose who to buy presents for is that they put all their names into a hat and then each student picks a random name from the hat. Each student buys a present for the student whose name they pick.

4. What's the expected number of students who have to buy presents for themselves? **Hint:** Let $X_i = 1$ if the ith student gets their own name and $X_i = 0$ otherwise.

   Answer: The probability a student gets their own name is 1/70. The expected number of students who have to buy presents for themselves is $E(X_1 + \ldots + X_{70}) = E(X_1) + \ldots + E(X_{70}) = 1/70 + \ldots + 1/70 = 1$.

5. **Extra Credit:** Consider a graph with nodes 1, 2, ..., 70 where there's a directed edge from i to j if the ith student bought a present for the jth student. This graph consists of node-disjoint cycles. What's the expected number of cycles? Note that if someone buys a present for herself, this counts as a cycle. Also, if the ith student buys a present for the jth student and the jth student buys a present for the ith student, this also counts as a cycle.

   Answer: Suppose the students draw their names in the following order: Student 1 picks first. At each subsequent step, either the student whose name was just picked goes next or, if they've already chosen, the student with the smallest number who hasn't already picked goes next. Note that the ordering will not change the result. Let $X_i = 1$ if the ith person who picks completes a cycle and 0 otherwise. Note that $E(X_i) = 1/(n-i+1)$ because when they pick there is one name they could pick out of the $n-i+1$ remaining names such that a cycle would be completed. Hence the expected number of cycles is:

$$E(X_1 + X_2 + \ldots X_{70}) = 1/70 + 1/69 + \ldots + 1/1 = 4.83\ldots \approx \ln 70 \ .$$