# CMPSCI 611: Advanced Algorithms

## Lecture 2: More Divide and Conquer

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# Outline

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# Divide and Conquer Methodology

- General Goal: Solve problem $P$ on an instance $I$ of "size" $n$.
- Divide & Conquer:
  1. Transform $I$ into smaller instances $I_1, \ldots, I_a$ each of "size" $n/b$
  2. Solve problem $P$ on each of $I_1, \ldots, I_a$ by recursion
  3. Combine the solutions to get a solution of $I$
- Example (Merge Sort): To sort $n$ numbers, divide into 2 sets of size $\frac{n}{2}$, sort each set, and merge.

# Analyzing Divide and Conquer Algorithms

Let $T(n)$ be running time of algorithm on instance of size $n$. Then

$$T(1) = \Theta(1), \qquad T(n) \leq aT(n/b) + O(n^\alpha)$$

where $O(n^\alpha)$ is time to create the subproblems and combine solutions.

## Theorem (Master Theorem)

*If $a, b, \alpha$ are constants,*

$$T(n) = \begin{cases} O(n^\alpha) & \textit{if } b^\alpha > a \\ O(n^{\log_b a}) & \textit{if } b^\alpha < a \\ O(n^\alpha \log n) & \textit{if } b^\alpha = a \end{cases}$$

- Example (Merge Sort): $a = b = 2$ and $\alpha = 1$. Therefore the running time is $O(n \log n)$.

# Outline

# First Attempt at Matrix Multiplication

Given two $n \times n$ matrices $A$ and $B$, multiply them together to get $C$:

$$c_{ij} = \sum_{k \in [n]} a_{ik} b_{kj}$$

Naive algorithm works in $O(n^3)$ time. Try Divide and Conquer...

- Divide $A$ and $B$ into four $n/2 \times n/2$ sub-matrices:

$$A = \begin{pmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{pmatrix} \qquad B = \begin{pmatrix} B_{11} & B_{12} \\ B_{21} & B_{22} \end{pmatrix}$$

- And note

$$C = \begin{pmatrix} A_{11}B_{11} + A_{12}B_{21} & A_{11}B_{12} + A_{12}B_{22} \\ A_{21}B_{11} + A_{22}B_{21} & A_{21}B_{12} + A_{22}B_{22} \end{pmatrix} = \begin{pmatrix} P_1 + P_2 & P_3 + P_4 \\ P_5 + P_6 & P_7 + P_8 \end{pmatrix}$$

  where $P_1 = A_{11}B_{11}$ and $P_2 = A_{12}B_{21}$ etc.

- Bad News: $T(n) = 8T(n/2) + \Theta(n^2)$ gives $T(n) = \Theta(n^3)$

# Strassen's Algorithm: General Technique + Creativity

- Along comes Volker Strassen in 1969...

# Strassen's Algorithm

Break the problem into 7 sub-problems:

$$\begin{aligned}
P_1 &= (A_{11} + A_{22})(B_{11} + B_{22}) \\
P_2 &= (A_{21} + A_{22})(B_{11}) \\
P_3 &= (A_{11})(B_{12} - B_{22}) \\
P_4 &= (A_{22})(-B_{11} + B_{21}) \\
P_5 &= (A_{11} + A_{12})(B_{22}) \\
P_6 &= (-A_{11} + A_{21})(B_{11} + B_{12}) \\
P_7 &= (A_{12} - A_{22})(B_{21} + B_{22})
\end{aligned}$$

Then

$$AB = \begin{pmatrix} P_1 + P_4 - P_5 + P_7 & P_3 + P_5 \\ P_2 + P_4 & P_1 - P_2 + P_3 + P_6 \end{pmatrix}$$

Good: $T(n) = 7T(\frac{n}{2}) + \Theta(n^2)$ gives $T(n) = \Theta(n^{2.81})$.

Improvements: $O(n^{2.376})$ by Coppersmith, Winograd 1990, $O(n^{2.3736})$ by Stothers 2010, $O(n^{2.3729})$ by Williams 2011, $O(n^{2.3728})$ by Le Gall 2014, $\ldots O(n^{2.371552})$ by Williams, Xu, Xu, and Zhou in 2024.

# Outline

# Finding Minimum Distance between Points on a Plane

Problem: Given $n$ distinct points $p_1, \ldots, p_n \in \mathbb{R}^2$, find

$$\text{minimum distance between any two points} = \min_{i \neq j} d(p_i, p_j)$$

How long does naive algorithm take? $O(n^2)$

We'll do it in $O(n \log n)$ steps.

For simplicity, assume no two points have the same $x$ or $y$ coordinate.

# Minimum Distance Algorithm

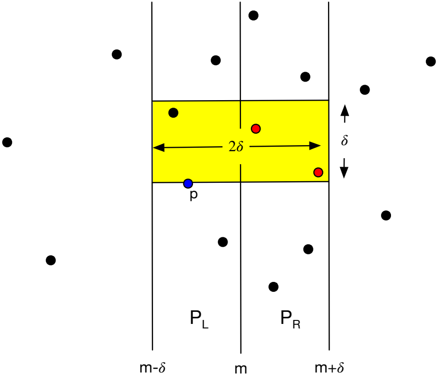
1. Divide points $P$ with a vertical line into $P_L$ and $P_R$ where
$$|P_L| = |P_R| = n/2$$
2. Recursively find minimum distance within $P_L$ and $P_R$:
$$\delta_L = \min_{p,q \in P_L : p \neq q} d(p,q)$$
$$\delta_R = \min_{p,q \in P_R : p \neq q} d(p,q)$$
3. Compute $\delta_M = \min_{p \in P_L, q \in P_R} d(p,q)$ and return
$$\min(\delta_L, \delta_R, \delta_M)$$

Note: If Step 3 takes $O(n^2)$ time, we get
$$T(n) \leq 2T(n/2) + O(n^2) \Longrightarrow T(n) = O(n^2)$$

If we can do Step 3 in $\Theta(n)$ time, we get $T(n) = O(n \log n)$.

# Making Step 3 Efficient

- Need to find $\min(\delta_L, \delta_R, \delta_M)$ where $\delta_M = \min_{p \in P_L, q \in P_R} d(p, q)$
- Suppose that the dividing line is $x = m$ and $\delta = \min(\delta_L, \delta_R)$
- Once we know $\delta$, only need $O(n)$ comparisons to find $\min(\delta, \delta_M)$
  1. Only compare $p = (p_1, p_2)$ to $q = (q_1, q_2)$ if
  $$p_2 \leq q_2 \leq p_2 + \delta \quad \text{and} \quad m - \delta < p_1, q_1 < m + \delta \ .$$
  2. Claim: Each point only needs compared with $\leq 6$ other points.



# Implementation details

- Need to identify which points to compare in $O(n)$ time
- Assume points are sorted by $y$-coordinate. Ensure list is passed to each recursion sorted.
- Given sorted list, it's easy to find the relevant points to compare
  1. Remove points whose $x$-coordinate differs from $m$ by more than $\delta$.
  2. Scan through rest from bottom to top, compare each point with the next 6 points in the list.
- Can find dividing line that splits $P_L$ and $P_R$ in $O(n)$ time.

# Proof of Claim

- All points in to be compared with $p$ lie in a $\delta \times \delta$ rectangle.
- Since each is at least $\delta$ away from the others, we can draw circles of radius $r = \delta/2$ around each and these circles do not overlap.
- The area of the intersection of a circle and the box is at least $\pi r^2/4$.
- Since the total area of the rectangle is $\delta^2$, the total number of points must be at most $\delta^2/(\pi r^2/4) = 16/\pi = 5.09\ldots < 6$.
- Better constants are possible but the exact constant isn't important.