CMPSCI 611: Advanced Algorithms

Lecture 5: Greedy Algorithms and Matroids

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Subset Systems

Definition

A *subset system* $S = (E, \mathcal{I})$ is a finite set E with a collection $\mathcal{I}$ of subsets E such that:

$$\text{if } A \in \mathcal{I} \text{ and } B \subset A \text{ then } B \in \mathcal{I}$$

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2. E is the edges of a graph and $\mathcal{I}$ is the acyclic subsets of edges
3. E is the edges of a graph and $\mathcal{I}$ are the matchings, i.e., subsets of edges such that no two edges share a vertex

Generic Problem and Greedy Algorithms

Problem Given a subset system $S = (E, \mathcal{I})$ and weight function $w : E \to \mathbb{R}^+$, find $A \in \mathcal{I}$ such that $w(A) = \sum_{e \in A} w(e)$ is maximized.

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Algorithm (Greedy)

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Terminology: Solution $A \in \mathcal{I}$ is a maximum if $w(A) \geq w(A')$ for all other $A' \in \mathcal{I}$. Solution $A \in \mathcal{I}$ is maximal if there doesn't exist $e \in E - A$ such that $A + e \in \mathcal{I}$.

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Let $E = \{e_1, e_2, e_3\}$, $\mathcal{I} = \{\{e_1, e_2\}, \{e_2, e_3\}, \{e_1\}, \{e_2\}, \{e_3\}, \{\}\}$, and $w(e_1) = 3, w(e_2) = 1$, and $w(e_3) = 4$. The greedy algorithm returns

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Example (Maximum Weight Matching)

E is the edges of a graph and $\mathcal{I}$ are the matchings. Greedy does not work.

Matroid Definition and Theorem

Definition

Subset system $(E, \mathcal{I})$ has the exchange property if

$$\forall A, B \in \mathcal{I} : (|A| < |B|) \implies (\exists e \in B - A \text{ such that } A + e \in \mathcal{I})$$

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Theorem

Given a subset system $(E, \mathcal{I})$, the following statements are equivalent:

1. *Greedy algorithm returns optimal solution for any weight function.*
2. *The subset system obeys the exchange property, i.e., it's a matroid.*

Matroid implies Greedy Algorithm is Optimal

- Proof by contradiction: Assume $(E, \mathcal{I})$ is a matroid and let

$$\text{greedy solution:} \quad A = \{e_1, e_2, \ldots, e_k\}$$
$$\text{optimal solution:} \quad B = \{f_1, f_2, \ldots, f_{k'}\} \quad \text{where } w(B) > w(A)$$

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- By the exchange property there exists $t \in [s]$ such that:

$$f_t \in \beta - \alpha \;\text{ with }\; \alpha + f_t \in \mathcal{I}$$

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- But then $w(f_t) \geq w(f_s)$ and hence $w(f_t) > w(e_s)$. This is a contradiction since greedy algorithm picked e_s rather than f_t

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- Let $m = |A|$ and $n = |E|$. Define weight function:

$$w(e) = \begin{cases} m+2 & \text{if } e \in A \\ m+1 & \text{if } e \in B - A \\ 1/(2n) & \text{otherwise} \end{cases}$$

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- Greedy algorithm returns A with weight at most $(m+2)m + 1/2$ but a better solution is B with weight at least $(m+1)^2$

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