CMPSCI 611: Advanced Algorithms

Lecture 6: Cardinality Theorem and Matroid Examples

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Outline

Summary of Matroid Results

The Problem

Definition

A *subset system* $S = (E, \mathcal{I})$ is a finite set E with a collection $\mathcal{I}$ of subsets of E such that if $A \in \mathcal{I}$ and $B \subset A$ then $B \in \mathcal{I}$.

Problem Given a subset system $S = (E, \mathcal{I})$ and weight function $w : E \to \mathbb{R}^+$, find $A \in \mathcal{I}$ such that $w(A) = \sum_{e \in A} w(e)$ is maximized.

Algorithm (Greedy)

1. $A = \emptyset$
2. *Sort elements of E by non-increasing weight*
3. *For each $e \in E$: If $A + e \in \mathcal{I}$ then $A \leftarrow A + e$*

Matroid Definition and Theorem

Definition

Subset system $(E, \mathcal{I})$ has the exchange property if

$$\forall A, B \in \mathcal{I} : (|A| < |B|) \implies (\exists e \in B - A \text{ such that } A + e \in \mathcal{I})$$

Definition

A subset system $(E, \mathcal{I})$ has the cardinality property if

$$\forall E' \subseteq E : (A, B \in \mathcal{I} \text{ maximal subsets of } E') \implies (|A| = |B|)$$

where we say $A \in \mathcal{I}$ is a maximal subset of E' if $A \subseteq E'$ and there doesn't exist $e \in E'$ such that $A + e \in \mathcal{I}$.

Theorem

Given a subset system $(E, \mathcal{I})$, the following statements are equivalent:

1. *Greedy algorithm returns optimal solution for any weight function.*
2. *The subset system obeys the exchange property.*
3. *The subset system obeys the cardinality property.*

Exchange Property implies Cardinality Property

- Suppose A, B are maximal subsets of $E' \subseteq E$. Need to show

$$|A| = |B|$$

- If $|B| > |A|$, the exchange property implies

$$\exists\ e \in B - A \text{ such that } A + e \in \mathcal{I}$$

- Note that $A + e$ would still be in E' since $e \in B \subseteq E'$.
- Thus A was not maximal in E'. Contradiction!

Cardinality Property implies Exchange Property

- Suffices to show that $(E, \mathcal{I})$ not a matroid implies there exists E' and $A, B \in \mathcal{I}$ such that $|A| < |B|$ and A, B are maximal in E'
- $(E, \mathcal{I})$ not a matroid implies that

$$\exists\ A, C \in \mathcal{I} \text{ such that } |A| < |C| \text{ and } \nexists\ e \in C - A \text{ with } A + e \in \mathcal{I}$$

- Define $E' = A \cup C$ and note that A is maximal in E'.
- There exists $B \in \mathcal{I}$ such that $C \subseteq B$ and B is maximal in E'.
- But $|B| \geq |C| > |A|$ as required.

Example 1

Theorem

The Maximum Weight Forest (MWF) subset system is a matroid.

Proof.

- Pick an arbitrary subset of edges $E' \subseteq E$.
- Let $n_1, \ldots, n_k$ be the number of nodes in the connected components.
- Any maximal acyclic subset of E' has size

$$(n_1 - 1) + (n_2 - 1) + \ldots + (n_k - 1) = n - k$$

  because a maximal acyclic subgraph of a connected graph on n_i nodes is a tree and has $n_i - 1$ edges.
- Cardinality Theorem implies that it's a matroid.

□

Example 2

Theorem

Let E be a set of directed edges and $\mathcal{I}$ be subsets such that no two edges in the same subset point to same node. This is a matroid.

Proof.

- For any $E' \subseteq E$, the number of edges in a maximal subset of E' is equal to the number of vertices pointed to in E'.
- Cardinality Theorem implies that it's a matroid.

$\square$