CMPSCI 611: Advanced Algorithms

Lecture 9: Dijkstra's Algorithm

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Shortest Paths

Let $G = (V, E)$ be a directed graph with weights $w : E \to \mathbb{R}^+$.

Definition

For path $p = (v_1, \ldots, v_k)$ be a path, define

$$w(p) = \sum_{i=1}^{k-1} w(v_i, v_{i+1}) \ .$$

The *shortest path* between u and v is

$$\delta(u, v) = \min\{w(p) : p \text{ is a path from } u \text{ to } v\}$$

if there is a path from u to v and ∞ otherwise.

Dijkstra's Warm-Up

Single-Source Problem: Given $s \in V$, find $\delta(s, v)$ for all $v \in V$.

Dijkstra's algorithm solves problem if all edges are non-negative:

- Maintains array $(d[v] : v \in V)$ where $d[v]$ will always be ∞ or the length of some path from s to v, not necessarily the shortest. Hence,

$$d[v] \geq \delta(s, v)$$

- Maintains a set of processed vertices R. We'll prove that for all $v \in R$:

$$d[v] = \delta(s, v)$$

Dijkstra's Algorithm

Algorithm

1. $d[s] = 0$ *and for* $s \neq v$:

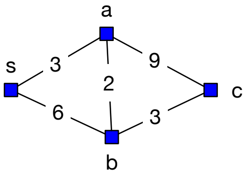
$$d[v] = w(s, v) \text{ if } (s, v) \in E \text{ and } \infty \text{ otherwise}$$

2. $R \leftarrow \{s\}$
3. *While* $|R| < |V|$:
   3.1 $u \leftarrow \operatorname{argmin}_{v \notin R} d[v]$
   3.2 $R \leftarrow R + u$
   3.3 *For each* $v \notin R$ *that is a neighbor of* u:

$$d[v] = \min(d[u] + w(u, v), d[v])$$

Running Time: $O(|V|^2)$ for simple implementation but can be improved.

Example



1. Step 1: $d[s] = 0, d[a] = 3, d[b] = 6, d[c] = \infty$, and $R = \{s\}$
2. Step 2: $d[s] = 0, d[a] = 3, d[b] = 5, d[c] = 12$, and $R = \{s, a\}$
3. Step 3: $d[s] = 0, d[a] = 3, d[b] = 5, d[c] = 8$, and $R = \{s, a, b\}$
4. Step 4: $d[s] = 0, d[a] = 3, d[b] = 5, d[c] = 8$, and $R = \{s, a, b, c\}$

Correctness of Algorithm

The correctness of the algorithm follows because a) $d[v]$ never increases, b) $d[v] \geq \delta(s, u)$ at all times, and c) appealing to the following lemma:

Lemma

When u is added to R, $d[u] = \delta(s, u)$

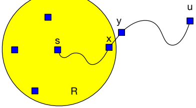
When u gets added to R, $d[u]$ is correct (1/2)

Let $d_u[v]$ be value of $d[v]$ just before u is chosen as minimum.

Lemma

For all u, $d_u[u] = \delta(s, u)$

- By contradiction: Let u be first vertex put in R with $d_u[u] > \delta(s, u)$
- Consider a shortest path from s to u. Let y be first vertex not in R. Note that y may or may not be u.



- Claim: $d_u[y] = \delta(s, y)$
  - Let x be the predecessor of y on the path. Note that $x \in R$.
  - $d_x[x] = \delta(s, x)$ by assumption that u is first bad vertex.
  - After iteration where x is added to R: $d[y] \leq \delta(s, x) + w(x, y)$
  - $\delta(s, x) + w(x, y) = \delta(s, y)$ since path included shortest path to y

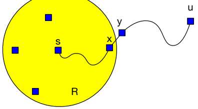
When u gets added to R, $d[u]$ is correct (2/2)

Let $d_u[v]$ be value of $d[v]$ just before u is chosen as minimum.

Lemma

For all u, $d_u[u] = \delta(s, u)$

- By contradiction: Let u be first vertex put in R with $d_u[u] > \delta(s, u)$
- Consider a shortest path from s to u. Let y be first vertex not in R. Note that y may or may not be u.



- Claim: $d_u[y] = \delta(s, y)$
- Since y lies on shortest path to u: $\delta(s, y) \leq \delta(s, u)$
- Putting above two lines together:

$$d_u[y] = \delta(s, y) \leq \delta(s, u) < d_u[u]$$

- If $y \neq u$: Contradiction because u was the next minimum and so

$$d_u[u] \leq d_u[y]$$

- If $y = u$: Contradicts $d_u[y] < d_u[u]$