CMPSCI 611: Advanced Algorithms

Lecture 10: Seidel's Algorithm

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Seidel's Algorithm

Problem: For an undirected, unweighted graph G, compute all distances.

Seidel's Algorithm is based on matrix multiplication and runs in time

$$O(\mu(n) \log n)$$

where $\mu(n)$ is the time to multiply two $n \times n$ matrices together. Recall

$$n^2 \leq \mu(n) \leq n^{2.3727}$$

Definition

Let M_G be the *adjacency matrix* of $G = (V, E)$, i.e., an $n \times n$ binary matrix where $M_G(i, j) = 1$ iff $(i, j) \in E$.

The G_2 graph

Definition

Given a undirected, unweighted graph $G = (V, E)$, define $G_2 = (V, E')$ where $(i, j) \in E'$ iff $\delta_G(i, j) \leq 2$.

Lemma

Let $P_G(i, j) = 1$ if $\delta_G(i, j)$ is odd and $P_G(i, j) = 0$ otherwise. Then,

$$\delta_G(i, j) = 2\delta_{G_2}(i, j) - P_G(i, j) \ .$$

Proof.

A path of length $2k$ in G corresponds to a path of length k in G_2. A path of length in $2k + 1$ in G corresponds to a path of length $k + 1$ in G_2. $\square$

Seidel's Algorithm

Algorithm (Seidel(M_G))

1. *compute* M_{G_2}
2. *if all non-diagonal entries of* $M_{G_2}(i,j)$ *are 1, return* D_G *where*

$$D_G[i,j] = \begin{cases} 0 & \textit{if } i = j \\ 1 & \textit{if } M_G(i,j) = 1 \\ 2 & \textit{otherwise} \end{cases}$$

3. *else:*
   3.1 *compute* $D_{G_2} = \text{Seidel}(M_{G_2})$
   3.2 *compute* P_G
   3.3 *return* $D_G = 2D_{G_2} - P_G$

Mystery Steps: How can we compute M_{G_2} and P_G efficiently?

Depth of Recursion

- The diameter of a graph G is the "longest shortest path",

$$\mathrm{diam}(G) = \max_{i,j} \delta_G(i,j)$$

- Note that if $\mathrm{diam}(G) \geq 3$:

$$\mathrm{diam}(G_2) \leq \frac{\mathrm{diam}(G)}{2} + \frac{1}{2} \leq \frac{2\mathrm{diam}(G)}{3}$$

- After recursing t steps, the diameter is at most

$$(2/3)^t \mathrm{diam}(G)$$

and so after $\log(n/2)/\log(3/2)$ steps, the diameter is at most 2.

Computing M_{G_2} via $M_G \times M_G$

Lemma

$$M_{G_2}(i,j) = \begin{cases} 1 & \text{if } i \neq j \text{ and } (M_G(i,j) = 1 \text{ or } M_G^2(i,j) > 0) \\ 0 & \text{otherwise} \end{cases}$$

Proof.

$M_G^2(i,j) = \sum_k M_G(i,k) M_G(k,j) = \#$ of length 2 paths from i to j. So there is an edge (i,j) in G_2 iff $(i,j) \in G$ or $M_G^2(i,j) > 0$. $\square$

Can compute M_{G_2} in $O(\mu(n))$ time.

Computing P_G via $D_{G_2} \times M_G$

P_G can be computed in $O(\mu(n))$ time...

Lemma

Let $X = D_{G_2} M_G$ *where* $D_{G_2}(i,j) = \delta_{G_2}(i,j)$. *Then,*

$$P_G(i,j) = 0 \iff \frac{X(i,j)}{\text{degree}_G(j)} \geq \delta_{G_2}(i,j)$$

where $\text{degree}_G(j)$ *is the number of edges incident to node* j *in graph* G.

Note that,

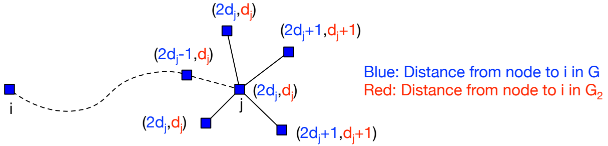
$$\frac{X(i,j)}{\text{degree}_G(j)} = \frac{\sum_k \delta_{G_2}(i,k) M_G(k,j)}{\text{degree}_G(j)} = \frac{\sum_{k:\text{neighbor of } j \text{ in } G} \delta_{G_2}(i,k)}{\text{degree}_G(j)}$$

Fix i and let $d_k = \delta_{G_2}(i,k)$, then we need to show:

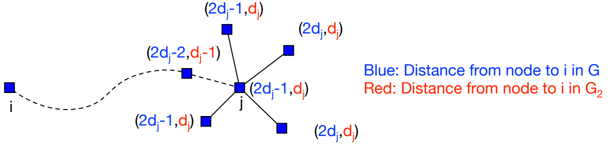
$$P_G(i,j) = 0 \iff (\text{average of } d_k \text{ over neighbors } k \text{ of } j) \geq d_j$$

Proof of Lemma

- If $P_G(i,j) = 0$, then $\delta_G(i,j) = 2d_j$



  - For all neighbors k note that $\delta_G(i,k)$ is either $2d_j - 1, 2d_j$, or $2d_j + 1$
  - Hence, each d_k is either d_j or $d_j + 1$
  - Therefore average d_k values is at least d_j
- If $P_G(i,j) = 1$, then $\delta_G(i,j) = 2d_j - 1$



  - For all neighbors k note that $\delta_G(i,k)$ is either $2d_j - 2, 2d_j - 1$, or $2d_j$
  - Hence, each d_k is either $d_j - 1$ or d_j
  - At least one neighbor has $\delta_G(i,k) = 2d_j - 2$ and $d_k = d_j - 1$
  - Therefore average d_k values is strictly less than d_j

Total Running Time

Algorithm (Seidel(M_G))

1. *compute* M_{G_2}
2. *if* $\forall i \neq j : M_{G_2}(i,j) = 1$, *return*

$$D_G(i,j) = \begin{cases} 0 & \textit{if } i \neq j \\ 1 & \textit{if } M_G(i,j) = 1 \\ 2 & \textit{if otherwise} \end{cases}$$

3. *else:*
   - 3.1 *compute* $D_{G_2} = \mathrm{Seidel}(M_{G_2})$
   - 3.2 *compute* P_G
   - 3.3 *return* $D_G = 2D_{G_2} - P_G$

Running Time: $O(\mu(n) \log n)$ since depth of recursion is $O(\log n)$ and each iteration takes $O(\mu(n))$ time.