CMPSCI 611: Advanced Algorithms

Lecture 14: Min-Cut

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From Last Time: Quicksort

Problem: Sort an array of distinct values $X = [x_1, \ldots, x_n]$

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2. *Construct new arrays $Y = [y_1, \ldots, y_k]$, $Z = [z_1, \ldots, z_{n-k-1}]$ where*

$$y < x < z \text{ for all } y \in Y, z \in Z$$

3. *Recursively sort Y and Z to get Y' and Z'*
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What's the expected number of comparisons performed in this algorithm?

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Lemma

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4. Because $H_n = 1 + 1/2 + 1/3 + \ldots + 1/n = O(\log n)$,

$$\mathbb{E}\left[\sum_{1 \le i < j \le n} Z_{ij}\right] \le \sum_{j=2}^{n} \sum_{k=2}^{n} \frac{2}{k} = n \cdot O(\log n) = O(n \log n)$$

□

Outline

Karger's Randomized Min-Cut Algorithm

Min-Cut Problem

Given an unweighted, multi-graph $G = (V, E)$, we want to partition V into V_1 and V_2 such that $|E \cap (V_1 \times V_2)|$ is minimized.

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- *Repeat until there are only 2 vertices remaining.*
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Let $|V| = n$ and $|E| = m$.

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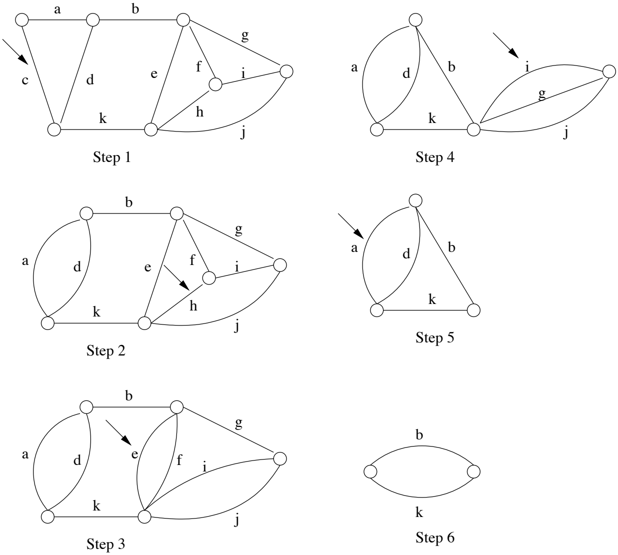

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$$\mathbb{P}\left[\cap_{1\leq i\leq n-2} A_i\right] = \mathbb{P}\left[A_1\right]\mathbb{P}\left[A_2|A_1\right]\ldots\mathbb{P}\left[A_{n-2}|\cap_{1\leq i\leq n-3} A_i\right]$$

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- Number of edges before i-th step if no edges across C have been contracted so far is at least

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- Number of edges before i-th step if no edges across C have been contracted so far is at least $(n - i + 1)k/2$ since there are $n - i + 1$ nodes remaining each with degree $\geq k$

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$$\mathbb{P}\left[\cap_{1 \leq i \leq n-2} A_i\right] \geq (1 - \frac{2}{n})(1 - \frac{2}{n-1})(1 - \frac{2}{n-2}) \ldots (1 - \frac{2}{3})$$

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Min-Cut Problem: Boosting the probability

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Proof.

- Because each repeat is independent,

$$\mathbb{P}\left[\text{always fails}\right] = \prod_{1 \leq i \leq \alpha n^2/2} \mathbb{P}\left[i\text{-th try fails}\right] \leq (1 - 2/n^2)^{\alpha n^2/2}$$

- Use fact $1 - x \leq e^{-x}$ for $x \geq 0$ and simplify.

$\square$