CMPSCI 611: Advanced Algorithms

Lecture 23: Linear Programming and Duality

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2. You make \$1 profit from Choco and \$6 profit from Choco Deluxe
3. Daily demand is 200 bars of Choco and 300 bars of Choco Deluxe
4. Your factory can produce at most 400 bars of chocolate a day
5. To maximize profit, what should you order from the factory?

Selling Chocolate: Linear Program

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Helpful to draw the "feasible region"...

Feasible Region

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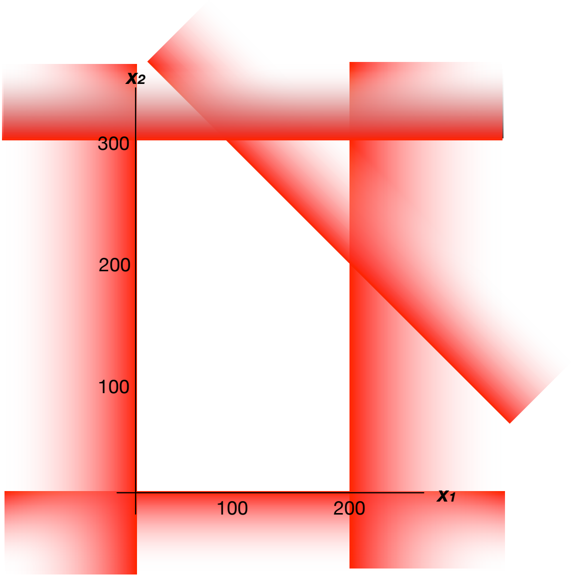

Concepts

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A linear program is *infeasible* if the constraints are so tight that it is impossible to satisfy all of them. E.g., $x \leq 1$, $x \geq 2$.

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Compute the objective function at each vertex... but this may take exponential time.

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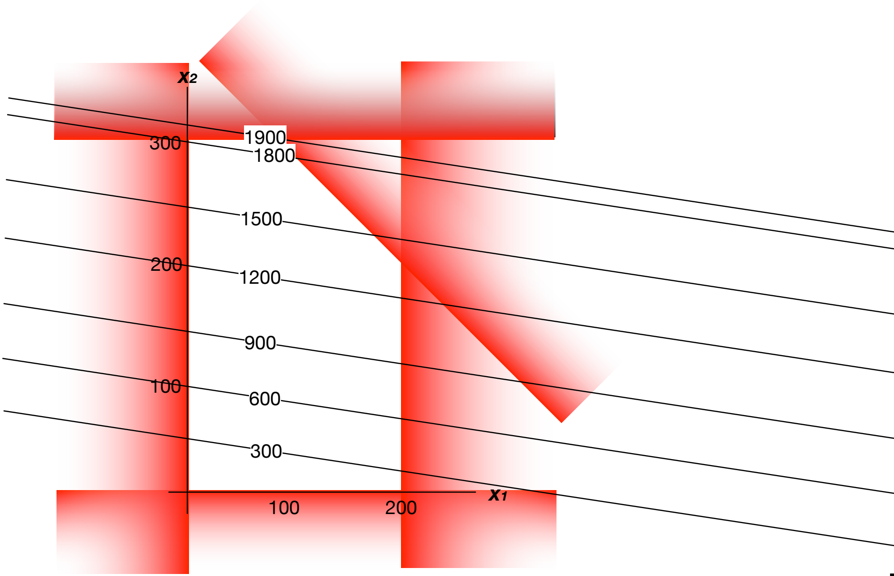
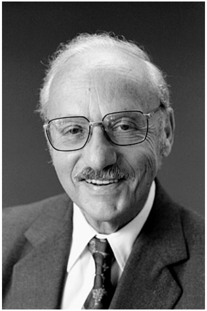


Better Algorithm: Simplex Algorithm

Simplex Algorithm was devised by George Dantzig in 1947...



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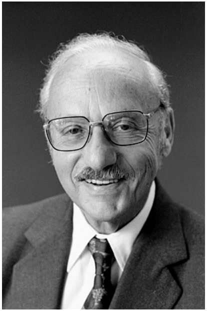

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Pick arbitrary vertex of the feasible region. Move to adjacent vertex with better objective value. If no such vertex exists, terminate.

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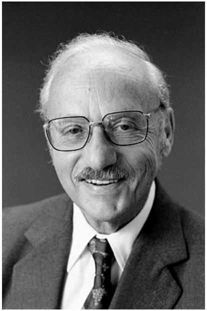

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Not known to be polynomial time but very quick in practice. Polynomial time algorithms do exist but are less used in practice.

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Need to visualize in 3D...

How do we know that a solution is optimal?

1. Suppose you friend claims that \$3100 is the optimum for

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and that this is achieved with $x_1 = 0, x_2 = 300, x_3 = 100$.

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2. Revisit constraints to certify that solution if optimal:

$$\begin{aligned} x_1 &\leq 200 && (1) \\ x_2 &\leq 300 && (2) \\ x_1 + x_2 + x_3 &\leq 400 && (3) \\ x_2 + 3x_3 &\leq 600 && (4) \end{aligned}$$

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3. Note that $0 \cdot \text{Eq. } (1) + 1 \cdot \text{Eq. } (2) + 1 \cdot \text{Eq. } (3) + 4 \cdot \text{Eq. } (4)$ is

$$x_1 + 6x_2 + 13x_3 \leq 3100$$

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4. But how did we come up with the coefficients $(0, 1, 1, 4)$?

Duality

- Back to simpler example: max $x_1 + 6x_2$ subject to

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- Adding one copy of Eq. (1) and seven copies of Eq. (2) gives

$$x_1 + 7x_2 \leq 2300$$

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  and so $x_1 + 6x_2 \leq 2300$ because $x_1, x_2 \geq 0$
- Adding five copies of Eq. (2) and one copy of Eq. (3) gives

$$x_1 + 6x_2 \leq 1900$$

More Duality

1. Trying to find multipliers that give good upper bound:

Multiplier	Constraint
y_1	$x_1 \leq 200$
y_2	$x_2 \leq 300$
y_3	$x_1 + x_2 \leq 400$

gives inequality $(y_1 + y_3)x_1 + (y_2 + y_3)x_2 \leq 200y_1 + 300y_2 + 400y_3$.

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2. If $y_1 + y_3 \geq 1$, $y_2 + y_3 \geq 6, y_1, y_2, y_3 \geq 0$, then an upper bound is

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3. Finding best such upper bound is new LP!

$$\text{Minimize: } 200y_1 + 300y_2 + 400y_3$$

   subject to

$$y_1 + y_3 \geq 1, \quad y_2 + y_3 \geq 6, \quad y_1, y_2, y_3 \geq 0$$

Duality in General

Primal and Dual Linear Programs:

Primal LP	Dual LP
$\max \mathbf{c}^T\mathbf{x}$	$\min \mathbf{y}^T\mathbf{b}$
$\mathbf{A}\mathbf{x} \leq \mathbf{b}$	$\mathbf{y}^T\mathbf{A} \geq \mathbf{c}^T$
$\mathbf{x} \geq \mathbf{0}$	$\mathbf{y} \geq \mathbf{0}$

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$\mathbf{A}\mathbf{x} \leq \mathbf{b}$	$\mathbf{y}^T\mathbf{A} \geq \mathbf{c}^T$
$\mathbf{x} \geq \mathbf{0}$	$\mathbf{y} \geq \mathbf{0}$

Theorem

Let $\mathrm{OPT}_{\text{primal}}$ *be optimal solution of Primal LP and let* $\mathrm{OPT}_{\text{dual}}$ *be optimal solution of Dual LP:*

$$\mathrm{OPT}_{\text{primal}} = \mathrm{OPT}_{\text{dual}}$$

and hence, any feasible solution of the dual LP upper bounds $\mathrm{OPT}_{\text{primal}}$*.*